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FOURIER TRANSFORM REPRESENTATION OF AN IDEAL LENS IN COHERENT OPTICAL SYSTEMS

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ABSTRACT

This document presents a mathematical analysis of the approximations required to obtain the Fourier transform representation of an ideal lens. An attempt is made throughout the paper to demonstrate the physical significance of the approximations and the variations from ideal results produced by neglected terms in the mathematical formulation. The approximations involved are considered in terms of the output signals in optical spectrum analyzer, optical imaging, and optical correlator systems.

CONTENTS

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INTRODUCTION

In recent years there has been a growing interest in the application of optical imaging techniques for the purpose of processing data signals. These efforts are largely based on the interpretation of optical imaging systems as spatial filters[1]. By introducing Fourier transform methods the relationbetween an object and its image has the same form as the relation between the input and output signals of an electrical system. Comparing the transform of the object and image, the imaging process (unity magnification is assumed here) can be described by the expression:

$$I(\omega_x, \omega_y) = T(\omega_x, \omega_y)\, O(\omega_x, \omega_y)$$

Image spectrum = Transfer function x Object spectrum

This expression has the same form as that for an electrical network except that the spectrums are two-dimensional. Since optical objects and images are two-dimensional, a Fourier transform must be taken with respect to each of the two spatial coordinates instead of the single time coordinate which appears in electrical signals.

It is the transfer function relation given above which leads to the spatial filter interpretation of optical imaging systems. The optical transfer function $T(\omega_x, \omega_y)$ is a characteristic of the optical elements in a system. An ideal imaging system would have a transfer function $T(\omega_x, \omega_y)$ which is constant over the object frequency range. In such an ideal system the image would be an exact replica of the object. The corresponding electrical system would have a flat frequency response over the bandwidth of the input signal.

It may appear at first thought that the transfer function notation is nothing more than an arbitrary selection of notation. However, in optical imaging systems, an object represented by a sinusoidal variation of light amplitude is imaged as a sinusoidal variation even in the presence of aberrations[2]. Aberration effects result in reduced contrast and a lateral shift of the sinusoidal image. Thus using sinusoidal test gratings it is possible, in theory, to experimentally determine the transfer function for a given optical system. In general, the optical transfer function can have complex values. The magnitude is related to the reduction in contrast, and the phase is related to the lateral shift of the image. In an actual system the transfer function will not have constant amplitude and phase as for the ideal imaging system described above.

Since the optical transfer function is determined by comparing the output image to the original object input, introducing any additional element into the optical system to vary the amplitude and/or phase transmission properties will change the optical transfer function. To utilize an optical system as a spatial filter in a fairly direct manner, it is necessary to know what amplitude and phase variations should be inserted and where they should be inserted. Otherwise, obtaining a particular transfer function for a spatial filter application would be a trial and error proposition. This implementation problem is solved for many cases of practical importance by the optical Fourier transform representation which is discussed in this report.

Within certain limitations the light amplitude distribution in the back focal plane of a lense is proportional to the two-dimensional Fourier transform of the light amplitude distribution of a two-dimensional object inserted on the front side of the lens. Within the range of validity for this optical Fourier transform representation, the transfer function is varied by a multiplicative factor represented by the amplitude and phase transmission properties of an element inserted into the back focal plane of the lens. For example, to set the transfer function equal to zero for a particular frequency component, the light passing through the corresponding point in the back focal plane of the lens is simply blocked.

The mathematical development of the optical Fourier transform representation presented in this report is intended to clarify the limitations and interpretation of the Fourier transform operation of a lens. The derivation is based on the Rayleigh-Sommerfeld diffraction formula and optical paths defined by geometrical ray tracing. As each limitation is introduced, an attempt is made to describe the effects on the accuracy of the optical Fourier transform representation. Such detailed consideration has been found lacking in available treatments[3] of the derivation and is the main purpose for the development presented in this report.

FOCAL PROPERTIES OF A LENS

To derive the formula for a focussed diffraction pattern, we will define the properties of an ideal lens. We will restrict our discussion to the case of an ideal lens and ignore the effects of lens aberrations, and diffraction at the edge of the lens. We assume that these effects can be taken into account by modifying our end result or by restricting the range of variables to a region in which our ideal assumptions are valid within experimental accuracies.

Our definition of an ideal lens will be based on the geometrical focussing properties shown in Figure 1. The properties assumed can be stated as follows:

1. The lens can be represented by a plane L perpendicular to the optical axis and all refraction takes place at this plane. This is the thin lens approximation which neglects the thickness of the lens.
2. The rays passing through the point O (intersection of the optical axis and the lens plane L) are called principal rays and will not be deviated.
3. All incident rays parallel to a principal ray will be focussed to the point at which the principal ray intersects the back focal plane F'. That is, the light reaching a point in the back focal plane F' at a distance $\rho = f \tan\theta$ (f is the focal length-distance between the planes L and F') from the optical axis is contributed by a principal ray making an angle θ with the optical axis plus all rays parallel to this principal ray.
4. If we construct a plane P perpendicular to a bundle of parallel incident rays, the optical path length will be the same along any of the parallel ray paths from P to the common point of focus in F'.

For any set of parallel rays, Figure 1, represents the projection of the parallel rays onto the plane through the optical axis and the principal ray as shown in Figure 2. In Figure 1 the distance ρ and ρ_1 are measured from the optical axis in the plane of the figure. As shown in Figure 2, these distances ρ and ρ_1 represent different quantities in the planes F and F'

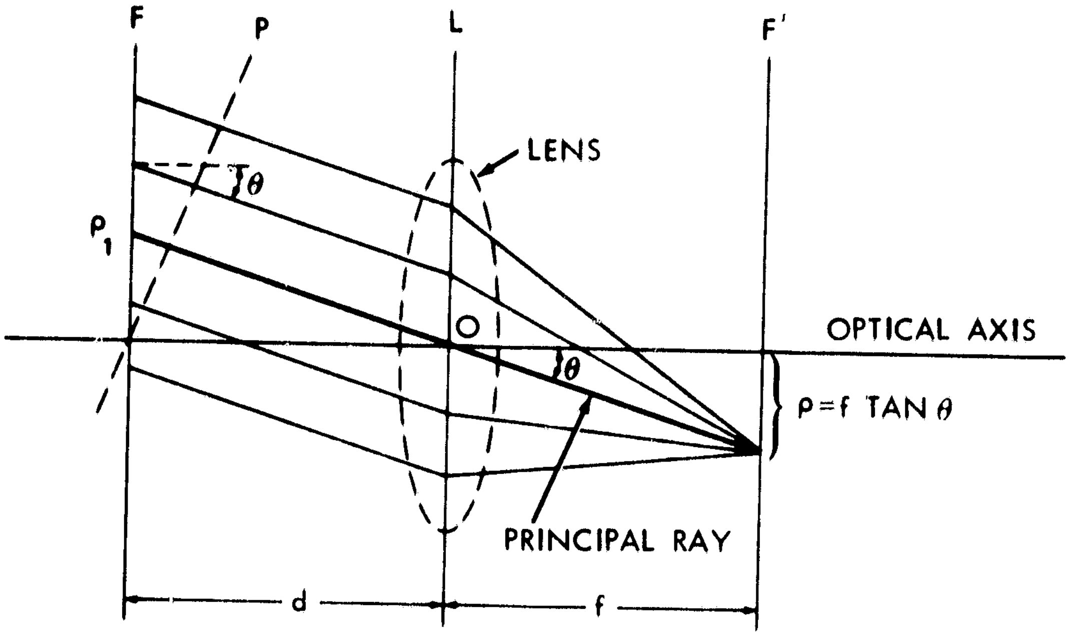


Figure 1—Ideal focussing of parallel rays.

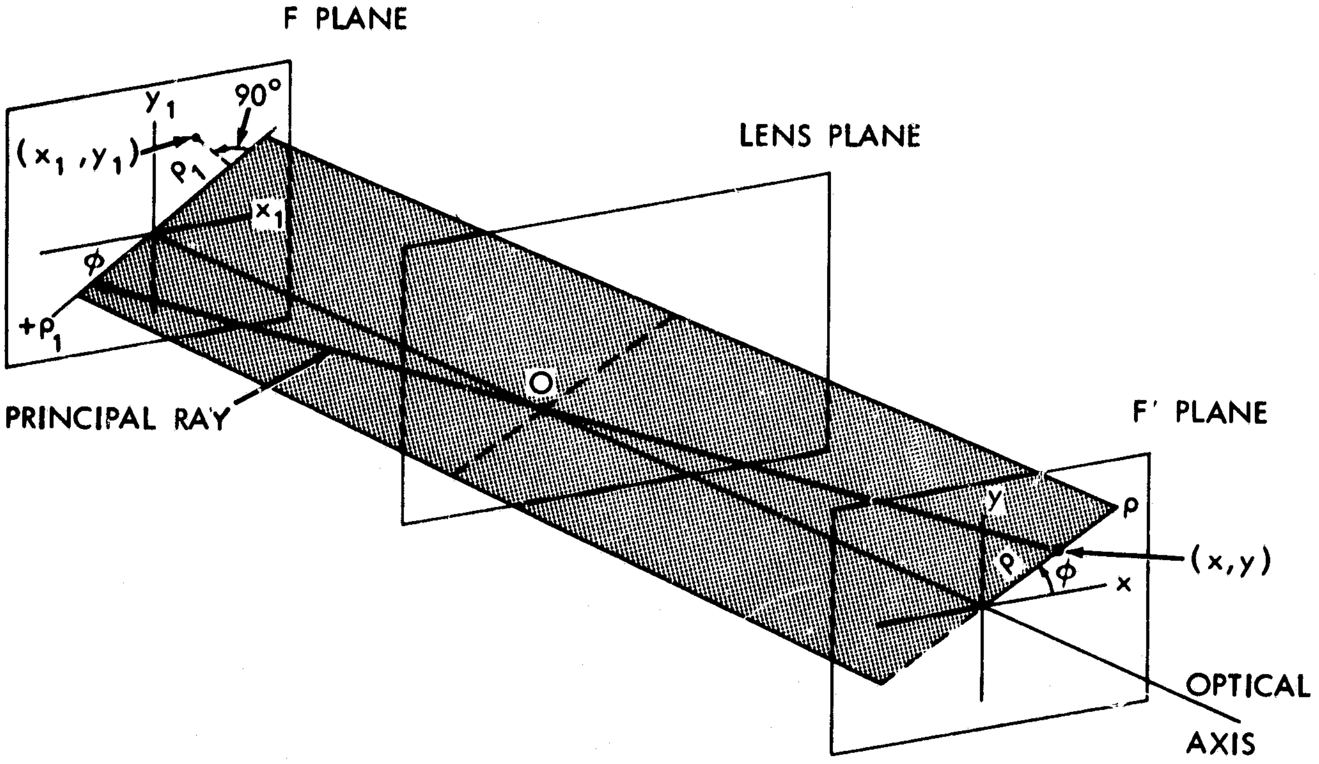


Figure 2—Difference between ρ and ρ_1.

respectively. The distance ρ in the back focal plane F′ is the distance from the optical axis to a point (x, y). In the plane F, ρ_1 is not the distance from the optical axis to a point (x_1, y_1) ; ρ_1 is the projection of this distance onto the axis defined by the intersection of the plane F with the plane through the optical axis and a point (x, y) in the plane F′ . Since the orientation of the ρ_1 axis will depend on the point (x, y) being considered in F′, ρ_1 will be a function of x, y, x_1 , and y_1 whereas ρ depends only on x and y . The difference in meaning of ρ and ρ_1 also appears when the algebraic sign is considered. In Figures 1 and 2, ρ is the distance from the optical axis to the point of focus and is a positive quantity regardless of where the point is located. On the other hand ρ_1 is a coordinate of the intersection of a ray with the F plane and we will use the sign convention that ρ_1 is positive above the axis and negative below the axis when drawn as in Figure 1 (rays sloping down to the right). In reference to a point (x, y) in F′ the positive ρ_1 axis will lie in the quadrant opposite to that of the point (x, y) .

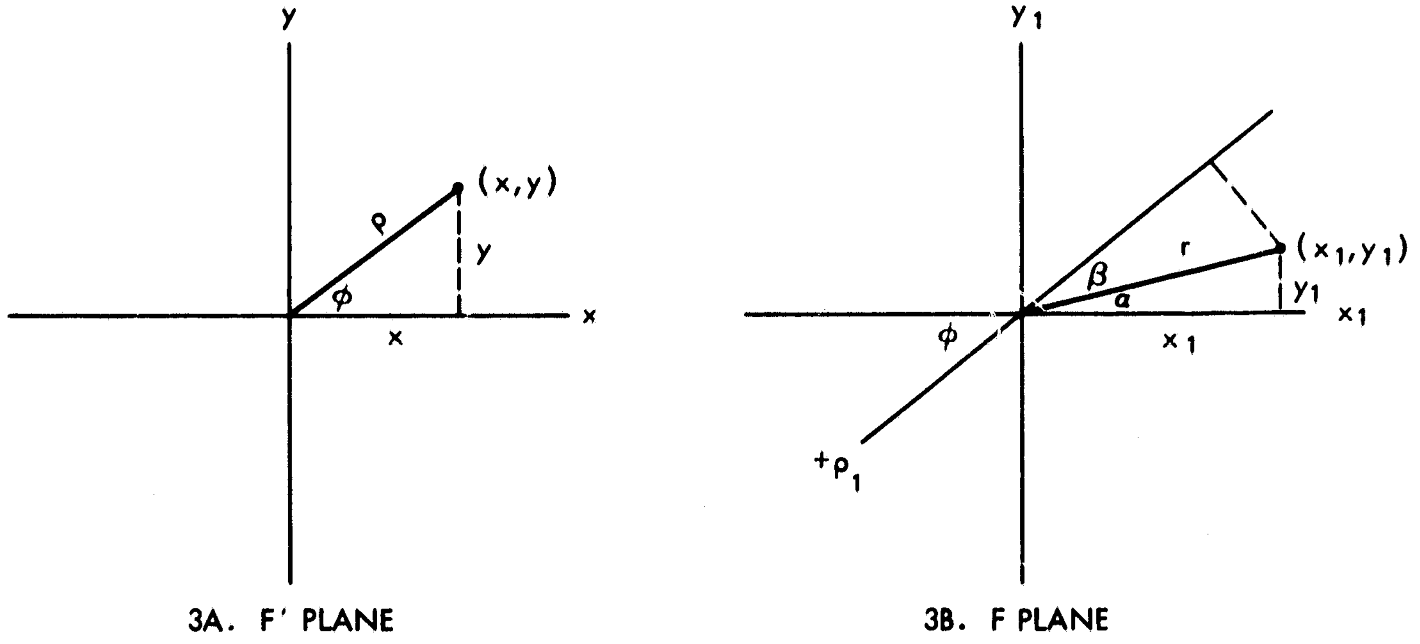


Figure 3—Geometry for ρ and ρ_1.

Figure 3 shows the geometry of the various lines in planes F and F′ and demonstrates the sign convention for ρ_1 . From the geometry of Figure 3A we find the relations:

$$\cos\phi = \frac{x}{\rho} \qquad \sin\phi = \frac{y}{\rho} \tag{1}$$

$$\rho = \left(x^2 + y^2\right)^{1/2}$$

From the geometry of Figure 3B we can determine ρ_1 for any point (x_1, y_1) as follows:

$$\beta = \alpha + \gamma , \qquad \cos \alpha = \frac{x_1}{r} , \qquad \sin \alpha = \frac{y_1}{r} , \qquad r = \left(x_1^2 + y_1^2\right)^{1/2}$$

$$\rho_1 = -\, r \cos \gamma = -\, r \cos (\beta - \alpha) = -\, r \{\cos \beta \cos \alpha + \sin \beta \sin \alpha\}$$

$$\text{or} \quad \rho_1 = -\, x_1 \cos \beta - y_1 \sin \beta$$

Substituting the relations for $\cos \beta$ and $\sin \beta$ obtained from 3A we can rewrite the expression for ρ_1 as

$$\rho_1 = \frac{-\, xx_1 - yy_1}{\rho} = \frac{-\, xx_1 - yy_1}{\left(x^2 + y^2\right)^{1/2}} \tag{2}$$

The dependence of ρ on x and y and of ρ_1 on x, y, x_1, and y_1 is explicit in equations (1) and (2) respectively. These results agree with our discussion in the last paragraph.

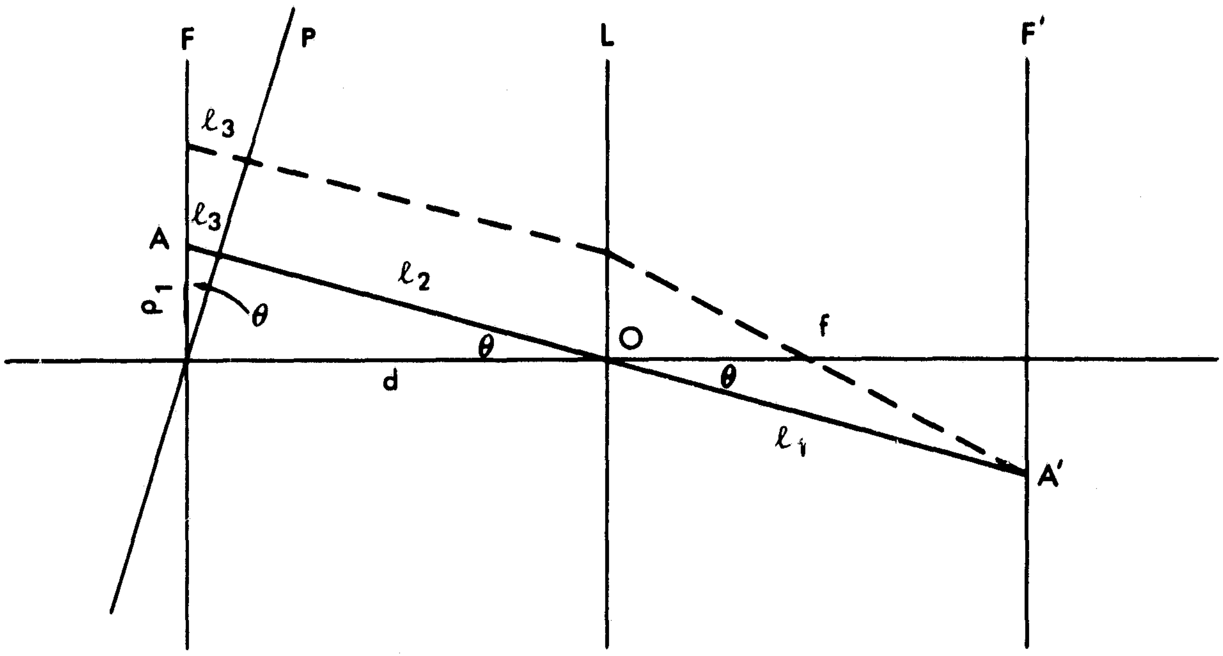


Figure 4—Geometry for optical path length.

We will now use Figure 4 to determine an expression for the optical path length from a point (x_1, y_1) in F to a point (x, y) in F'. Figure 4 shows a principal ray AA' and a representative parallel ray (dotted). The plane P is perpendicular to the incident rays. Since we are neglecting the thickness of the lens in our discussion, the optical path length r for the principal ray AA$'$ will be given by the geometrical length:

$$r = \ell_1 + \ell_2 + \ell_3 \tag{3}$$

From the geometry of Figure 4, ℓ_1 is given by the expression

$$\ell_1 = \left(f^2 + \rho^2\right)^{1/2} \tag{4}$$

By similar triangles, we find that ℓ_2 is given by

$$\frac{\ell_2}{d} = \frac{f}{\ell_1} \qquad \text{or} \qquad \ell_2 = \frac{df}{\ell_1} \tag{5}$$

For the principal ray AA$'$ the optical path length from the plane P to the point A$'$ in the back focal plane F$'$ is $\ell_1 + \ell_2$. Using equations(4) and (5) this length is given as

$$\ell_1 + \ell_2 = \ell_1 + \frac{df}{\ell_1} = \frac{\ell_1^2 + df}{\ell_1} = \frac{f^2 + \rho^2 + df}{\left(f^2 + \rho^2\right)^{1/2}}$$

By the fourth of our assumptions for an ideal lens the optical path length from the plane P to the focus point A$'$ is the same for all the rays parallel to the principal ray AA$'$ (note that $\ell_1 + \ell_2$ does not depend on ρ_1). Therefore the expression found for $\ell_1 + \ell_2$ holds for every parallel ray and the expression for the optical path length r can be written as

$$r = \ell_1 + \ell_2 + \ell_3 = \frac{f^2 + \rho^2 + df}{\left(f^2 + \rho^2\right)^{1/2}} + \ell_3 \tag{6}$$

The term ℓ_3 remains to be determined and by comparison of the principal ray and the representative ray in Figure 4 it should be obvious that this term will not be the same for all parallel rays. From the right triangle with ℓ_3 as a leg, ℓ_3 can be expressed as

$$\ell_3 = \rho_1 \sin\theta$$

However, $\sin\theta = \frac{\rho}{(f^2 + \rho^2)^{1/2}}$ and the expression for ℓ_3 can be rewritten as

$$\ell_3 = \frac{\rho\rho_1}{\left(f^2 + \rho^2\right)^{1/2}} \tag{7}$$

Substituting for ℓ_3 in the expression for the optical path length r we obtair

$$r = \frac{f^2 + df + \rho^2 + \rho\rho_1}{\left(f^2 + \rho^2\right)^{1/2}} \tag{8}$$

We have previously derived expressions for ρ and ρ_1 as given by equations (1) and (2). Substituting for ρ and ρ_1 in equation (8) we obtain

$$r = \frac{f^2 + x^2 + y^2 + df - xx_1 - yy_1}{\left(f^2 + x^2 + y^2\right)^{1/2}} \tag{9}$$

This expression gives the optical path length from any point (x_1, y_1) in a plane F (a distance d in front of the lens) to a point (x, y) in the back focal plane F' . To facilitate further discussion we will write this expression as

$$r = R(x, y) - \alpha x_1 - \beta y_1 \tag{10}$$

where:

$$R(x, y) = \frac{f^2 + df + x^2 + y^2}{\left(f^2 + x^2 + y^2\right)^{1/2}}$$

$$\alpha = \frac{x}{\left(f^2 + x^2 + y^2\right)^{1/2}}$$

$$\beta = \frac{y}{\left(f^2 + x^2 + y^2\right)^{1/2}}$$

To summarize our results at this point, we can consider a point A in the plane F as shown in Figure 5. Assuming that light is radiated in all directions from the point A, we will consider the portion of light propagated in directions at an angle θ with respect to the normal n to the plane F . The light rays representing these directions will form the surface of a cone of half angle θ as shown in Figure 5. For each of these rays a parallel principal ray can be drawn and each principal ray will make an angle θ with the optical axis. Thus for each ray at an angle θ with respect to the normal to F at the point A, we can apply the results derived above. That is, each ray is focussed to a point on the ring of radius $\rho = f \tan\theta$ where the corresponding principal ray intersects the back focal plane F'. The optical path length from A to each point on the ring is given by equation (10). Since this holds for any point A, we can state the general focal properties of our ideal lens as:

1. The light radiated from all points on F in directions at an angle θ with respect to the normal to F is focussed into a ring of radius $\rho = f \tan\theta$ in the back focal plane F'.
2. The optical path length r from any point (x_1, y_1) on the plane F to any point (x, y) in the back focal plane F' is given by equation (10).

In these statements the plane F is a plane perpendicular to the optical axis at a distance d in front of the lens.

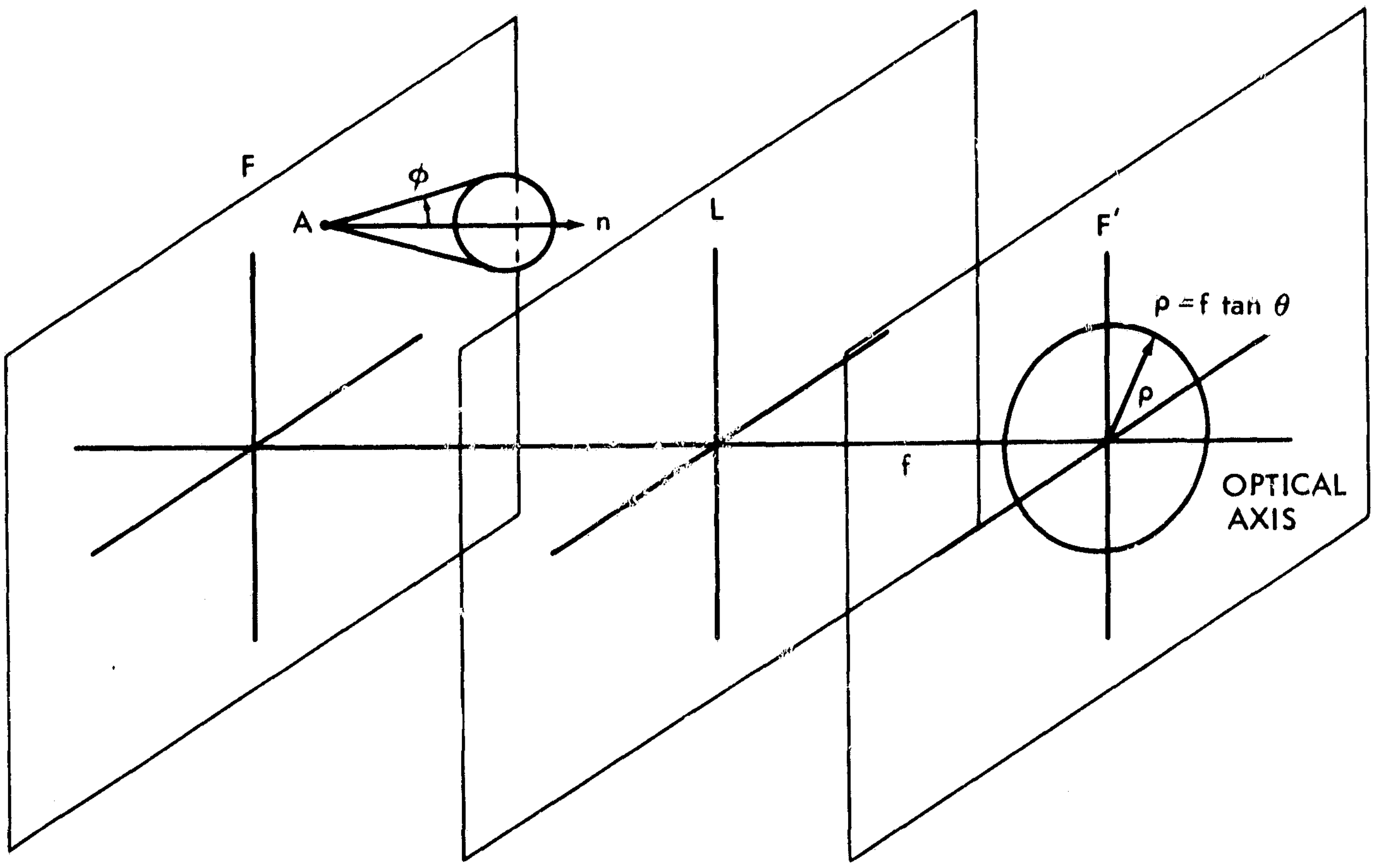


Figure 5—Focussing of a cone of light.

Anticipating the derivations in the next section, the relation between ρ and θ specified by the first focal property above can also be expressed in terms of $\cos\theta$ as

$$\cos\theta = \frac{f}{\left(f^2 + \rho^2\right)^{1/2}} = \frac{f}{\left(f^2 + x^2 + y^2\right)^{1/2}} \tag{11}$$

This expression can be derived from the geometry of the figures in this section or can be derived from the equation in terms of $\tan\theta$ by applying trigonometric identities.

FOCUSSED DIFFRACTION PATTERN

Since we will consider only light distributions on plane surfaces, we can use the Rayleigh-Sommerfeld diffraction formula (see Appendix). In rectangular coordinates this diffraction formula has the form:

$$A(x, y, z) = \frac{-1}{2\pi}\iint A'(x_1, y_1)\,\frac{e^{ikr}}{r}\left[ik - \frac{1}{r}\right]\cos\theta\, dx_1\, dy_1 \qquad (12)$$

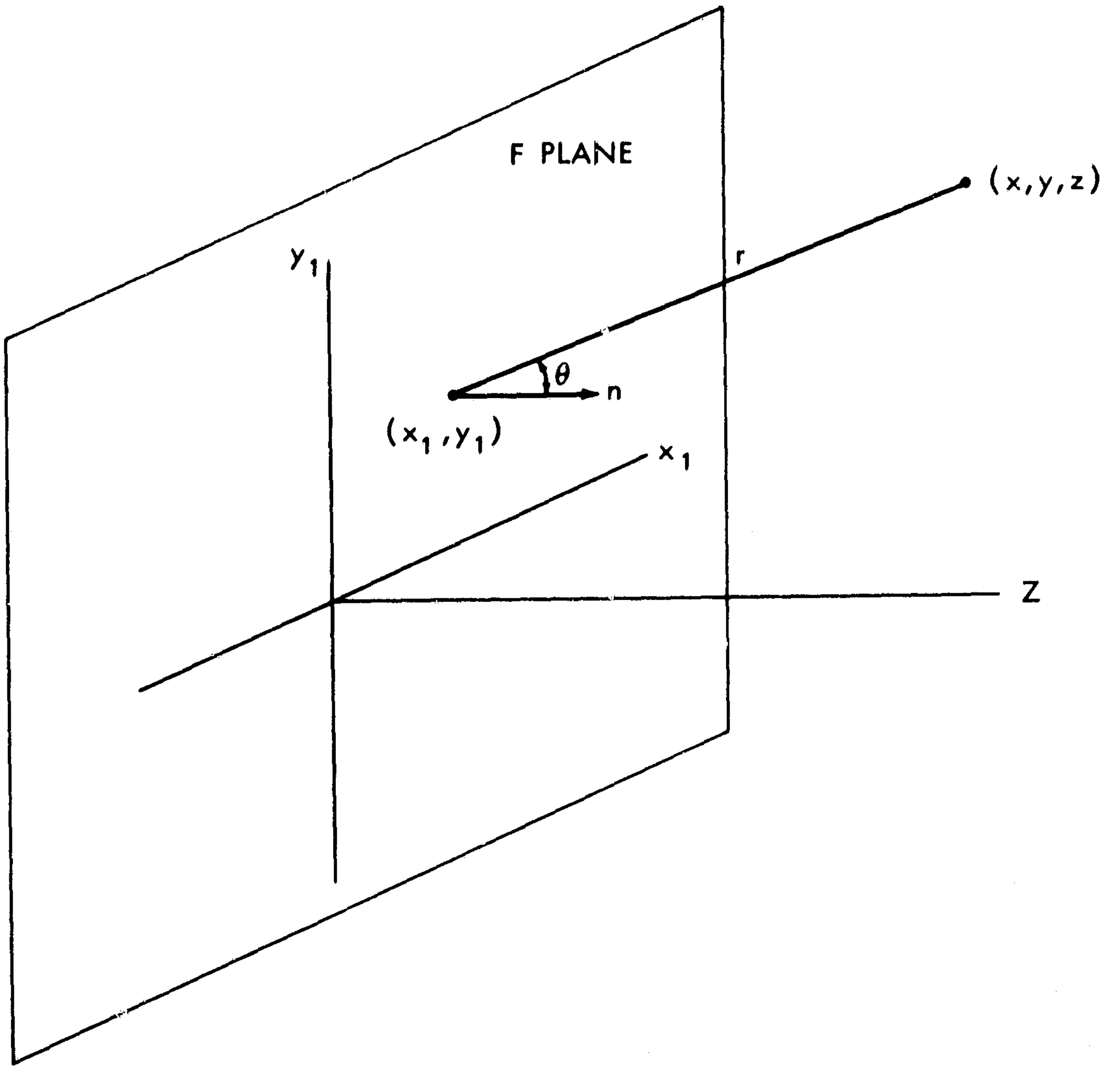


Figure 6—Relation between points (x, y, z) and (x_1, y_1) in diffraction formula.

This formula gives the complex light amplitude $A(x,y,z)$ at any point in space $(z > 0)$ due to a monochromatic coherent light distribution $A'(x_1, y_1)$ given for every point (x_1, y_1) in a plane F . Referring to Figure 6, the terms in the diffraction formula are defined as follows:

1. $A'(x_1, y_1)$ is the complex amplitude of monochromatic light given for all points (x_1, y_1) in a plane F located at $z = 0$.
2. $A(x, y, z)$ is the complex amplitude of light produced by $A'(x_1, y_1)$ at a point (x, y, z) in space $(z \geq 0)$.

3. r is the distance from a point (x_1, y_1) in plane F to the point (x, y, z) .

4. θ is the angle between r and n where r is directed from (x_1, y_1) to (x, y, z) and n is the normal to the plane F at (x_1, y_1) in the direction of the positive z axis. The term $\cos\theta$ is usually referred to as the obliquity factor.

5. $k = \frac{2\pi}{\lambda}$, where λ is the wavelength of the monochromatic light.

In general each of the three vector components of the electromagnetic field representing the light distribution must be determined by the diffraction formula. For our discussion we will consider the light amplitude distribution as a scalar which requires only one equation[4]. In practice this is permissible if polarization affects can be neglected. Thus we will define the light amplitude such that the square of its absolute magnitude gives the intensity which is a measurable quantity.

We can simplify the diffraction formula immediately by considering the relative magnitudes of the terms inside the brackets:

$$\left[ik - \frac{1}{r}\right] = i\,\frac{2\pi}{\lambda} - \frac{1}{r}$$

For wavelengths λ as long as 100 microns (far infrared) the first term is relatively large (600) while for r larger than 1cm. the second term is less than one. For visible light λ is much less than 100 microns (0.4 to 0.7 microns) and k is of the order of 10^5. Since our discussion (as in most cases in optics) will deal only with r greater than one centimeter, the $\frac{1}{r}$ term is negligible and can be dropped without any appreciable effect on accuracy. The diffraction formula equation (12), can therefore be written as:

$$A(x, y, z) = \frac{-i}{\lambda}\iint A'(x_1, y_1)\,\frac{e^{ikr}}{r}\,\cos\theta\,dx_1\,dy_1 \qquad (13)$$

where the constant factor ik has been taken outside the integral.

The obliquity factor $\cos\theta$ is a weighting factor which accounts for the difference in the amount of light radiated in different directions. Since $\cos\theta$ has a maximum value of one at θ equal to zero, this factor has a maximum value of one for light contributions propagated normal to the signal plane and drops off as the angle with respect to the surface normal increases. Referring to Figure 6, if we assume the light from a point (x_1, y_1) contributing to the light at the point (x, y, z) travels the straight line r , this line is a light ray at an angle θ to the normal n . In the previous section, we showed that through the focal property of an ideal lens this angle is a constant for all light contributing to a point (x, y) in the back focal plane and that x, y and θ are related by the expression

$$\cos\theta = \frac{f}{\left(f^2 + x^2 + y^2\right)^{1/2}} \qquad (11)$$

In other words, an ideal lens focusses light of constant obliquity factor into a ring of radius $(x^2 + y^2)^{1/2}$ specified for a given θ by the above expression.

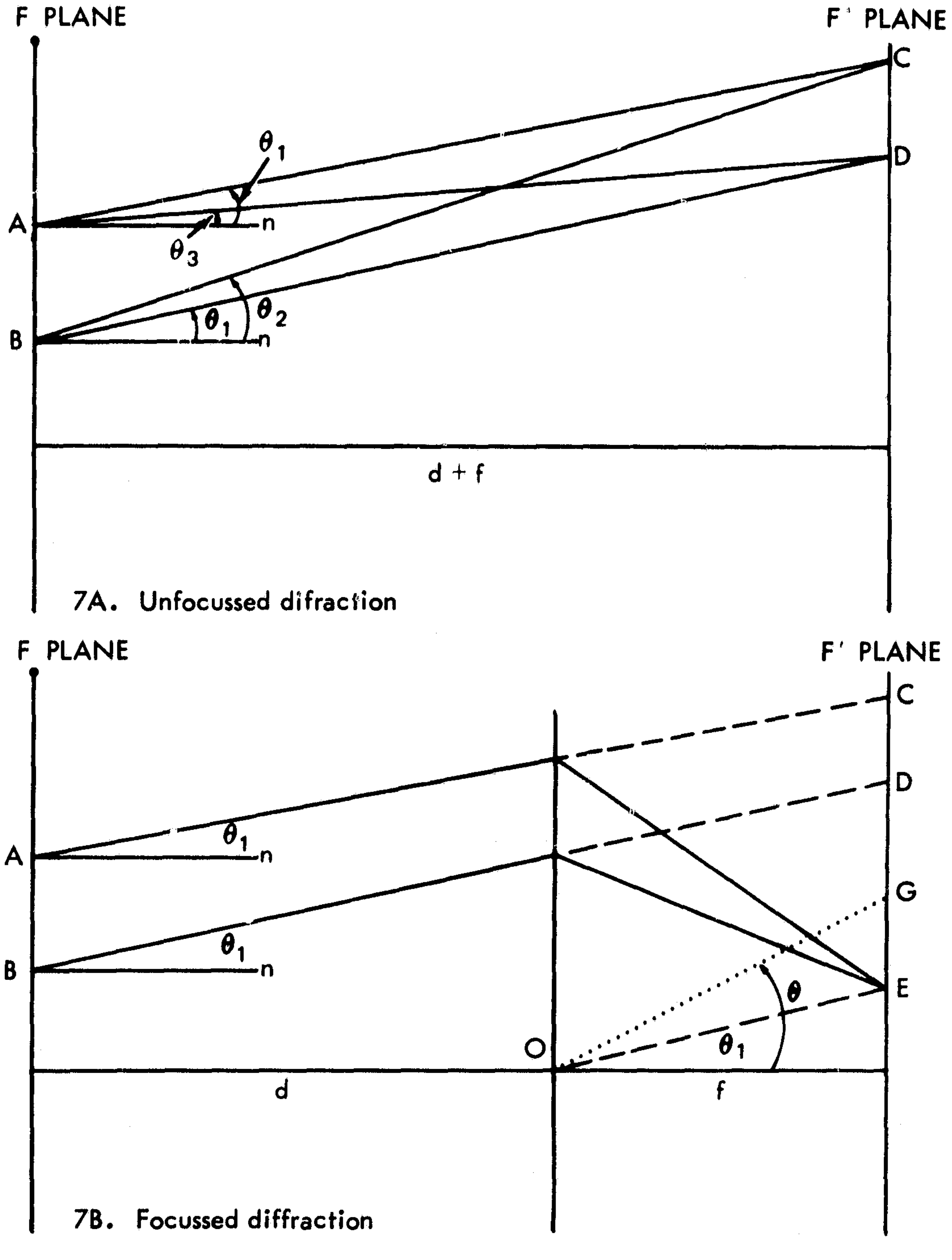


Figure 7—Comparison of diffraction with and without focussing.

The significance of this focal effect can be seen by comparing the two diagrams in Figure 7. In 7A, the points A and B are sample points in the F plane and the point C and D are sample points in a parallel plane at $z = d + f$. If we consider the point C, we note that the paths AC and BC have obliquity factors of $\cos\theta_1$ and $\cos\theta_2$ respectively. From this example it is obvious that for a point such as C (or D) the obliquity factor will depend on the location of the contributing point (x_1, y_1) in F. Likewise if we consider the point A in F, we can note that the paths AC and AD have obliquity factors of $\cos\theta_1$ and θ_3 respectively. This indicates that the obliquity factor also depends on the location of the point (x, y, z) . Since determining the obliquity factor is included in the derivation of the diffraction formula, we will give it here without proof for $z = d + f$ as shown in 7A:

$$\cos\theta = \frac{d + f}{\left[(x - x_1)^2 + (y - y_1)^2 + (d + f)^2\right]^{1/2}} \qquad (14)$$

This expression includes the coordinates of both the point (x_1, y_1) in the source plane and the point (x, y, z) at which the diffracted light amplitude is to be found. The expression for the general case will have a z in place of $(d + f)$ which was used for the special case of Figure 7A. In the diffraction formula the obliquity factor appears under the integral sign since x_1 and y_1 are the variables of integration and these terms appear in the obliquity factor as given by equation (14).

Let us now consider the case of focussed diffraction as illustrated in 7B. In 7B only the rays AC and BD of 7A are considered and as indicated by the angle θ_1, AC and BD are parallel rays. The dotted portion of these rays indicates the path of light followed in 7A. Due to refraction by the lens these paths are changed to those focussed to the point E . Now when we consider a point such as E we find that the obliquity factor $\cos\theta_1$ is the same for points A and B and therefore independent of the coordinates (x_1, y_1) of the point in F . If we consider any other point G in F$'$ we recall that to contribute to a point G a ray must be parallel to the principal ray OG. Rays parallel to OG will have an obliquity factor $\cos\theta$ different from $\cos\theta_1$ for the point E . Thus the obliquity factor does depend on the location of the point (x, y) in the back focal plane. The obliquity factor for the case of a focussed diffraction pattern is given by equation (11) as $\cos\theta = \frac{f}{(f^2 + x^2 + y^2)^{1/2}}$ and does not depend on the coordinates x_1 and y_1 .

Since the obliquity factor for the focussed diffraction pattern is independent of the integration variables x_1 and y_1 , this factor can be taken outside the integral and we can write equation (13) as

$$A(x, y) = \frac{-if}{\lambda\left(f^2 + x^2 + y^2\right)^{1/2}} \iint A'\left(x_1, y_1\right) \frac{e^{ikr}}{r}\, dx_1\, dy_1 \qquad (15)$$

where $A(x, y)$ now represents the complex light amplitude at a point (x, y) in the back focal plane of a lens.

To complete our discussion we will now consider the term r which was defined as the distance from the contributing point to the point of interest. In Figure 6 this distance r is measured along the straight line from (x_1, y_1) to (x, y, z) . In Figure 7B, the light traveling from A to E does not follow a straight line due to refraction at the lens plane L . We can assume that the effects due to the length of the refracted path are the same as traveling an equivalent distance in a straight line, and the r in the diffraction formula can be interpreted as the optical path length which we determined in the previous section. Thus e^{ikr} represents the change in phase over an optical length r , and $\frac{1}{r}$ is an attenuation factor which decreases the amplitude contribution as the optical path length increases.

Substituting the optical path length expression for r , as given by equations (9) and (10) the equation (15) becomes

$$A(x, y) = \frac{-if}{\lambda\left(f^2+x^2+y^2\right)^{1/2}} \iint \frac{A'\left(x_2, y_1\right) e^{ik[R(x,y)-\alpha x_1-\beta y_1]}}{\left[\frac{f^2+x^2+y^2+df-xx_1-yy_1}{\left(f^2+x^2+y^2\right)^{1/2}}\right]} dx_1\, dy_1 \tag{16}$$

The terms $(f^2+x^2+y^2)^{1/2}$ cancel and $e^{ikR\,(x,y)}$ can be taken outside the integral to give

$$A(x, y) = \frac{-if}{\lambda} e^{ikR(x,y)} \iint \frac{A'\left(x_1, y_1\right) e^{-i2\pi[\alpha x_1/\lambda+\beta y_1/\lambda]}}{f^2+x^2+y^2+df-xx_1-yy_1} dx_1\, dy_1 \tag{17}$$

We can now introduce the new variables p and q defined as

$$p = \frac{\alpha}{\lambda} = \frac{x}{\lambda\left(f^2+x^2+y^2\right)^{1/2}} \tag{18a}$$

$$q = \frac{\beta}{\lambda} = \frac{y}{\lambda\left(f^2+x^2+y^2\right)^{1/2}} \tag{18b}$$

and factor $f^2 + df$ from the denominator of the integral of equation (17) to obtain:

$$A(x, y) = \frac{-i}{\lambda(f+d)} e^{ikR(x,y)} \iint \frac{A'\left(x_1, y_1\right) e^{-i2\pi(px_1+qy_1)}}{1+\frac{x\left(x-x_1\right)+y\left(y-y_1\right)}{f(f+d)}} dx_1\, dy_1 \tag{19}$$

FOURIER TRANSFORM APPROXIMATION

We can now use the algebraic identity:

$$\frac{1}{1+\frac{M}{N}} = 1 - \frac{1}{1+\frac{N}{M}}$$

to obtain:

$$\frac{1}{1+\dfrac{x(x-x_1)+y(y-y_1)}{f(f+d)}} = 1 - \frac{1}{1+\dfrac{f(f+d)}{x(x-x_1)+y(y-y_1)}}$$

Introducing this identity we can write equation (19) as

$$A(x, y) = \frac{-i}{\lambda(f+d)} e^{ikR(x,y)} \iint \left[1 - \frac{1}{1+\dfrac{f(f+d)}{x(x-x_1)+y(y-y_1)}} \right] A'\left(x_1, y_1\right) e^{-i2\pi(px_1+qy_1)} dx_1 \, dy_1 \qquad (20)$$

We can rewrite equation (20) with an integral for each term in the bracket which gives

$$\begin{aligned} A(x, y) = {} & \frac{-i}{\lambda(f+d)} e^{ikR(x,y)} \iint A'\left(x_1, y_1\right) e^{-i2\pi(px_1+qy_1)} dx_1 \, dy_1 \\ & + \frac{i}{\lambda(f+d)} e^{ikR(x,y)} \iint \frac{A'\left(x_1, y_1\right) e^{-i2\pi(px_1+qy_1)}}{1+\dfrac{f(f+d)}{x(x-x_1)+y(y-y_1)}} dx_1 \, dy_1 \end{aligned} \qquad (21)$$

By restricting the maximum values (aperture limits) of x, y, x_1, y_1, the second integral of equation (21) can be made negligible compared to the first since the denominator of the integrand can be made large. This approximation will be discussed in more detail later; here we will simply assume that it is possible to neglect the second integral.

The diffraction formula can then be written approximately as:

$$A(x, y) = \frac{-i\, e^{ikR(x,y)}}{\lambda(f+d)} F(p, q) \qquad (22)$$

where $F(p,q)$ is the two-dimensional Fourier transform of $A'(x_1,y_1)$:

$$F(p, q) = \iint A'\left(x_1, y_1\right) e^{-i2\pi(px_1+qy_1)} dx_1 \, dy_1$$

If we were to measure this light distribution, we would measure the intensity which is the square of the magnitude of the complex amplitude $A(x,y)$. This intensity is given as:

$$I(x, y) = A(x, y) A^*(x, y) = \frac{1}{[\lambda(f+d)]^2} F(p, q) F^*(p, q)$$

$$\text{or} \quad I(x, y) = \frac{|F(p, q)|^2}{[\lambda(f+d)]^2} \tag{23}$$

Thus the intensity in the back focal plane of a lens (within the limits to be determined for the approximation made) is given by the square of the magnitude of the Fourier transform of the light amplitude in the plane F .

If our aperture restrictions in the back focal plane limit the maximum values of x and y so that the phase variations due to the exponential term in front of the integral in equation (22) can be considered constant, we can write

$$A(x, y) = KF(p, q) \tag{24}$$

where K is a complex constant given by

$$K = \frac{-ie^{ikR(x,y)}}{\lambda(f+d)}$$

Thus within the range of x and y for which $e^{ikR(x,y)}$ can be assumed constant (i.e. negligible phase variation) the amplitude distribution in the back focal plane is proportional to the Fourier transform of the light amplitude distribution in the plane F. This relationship requires tighter restrictions on x and y than our previous approximation. In terms of spectrum analysis in the back focal plane, this relation is not important since only intensity can be measured. However, in cascaded lens systems, the Fourier transform relation between amplitudes allows each pair of lenses to be accounted for by a double Fourier transform operation. The advantages of such a relation will be demonstrated in a later section.

FOURIER COMPONENTS (p, q) AND FOCAL PLANE COORDINATES (x, y)

In our previous discussions we have expressed the amplitude distribution $A(x,y)$ in terms of the Fourier transform of $A'(x_1, y_1)$ (refer to equation (22)). However, the Fourier transform coordinates are p and q which we defined by equations (18) as

$$p = \frac{x}{\lambda\left(f^2 + x^2 + y^2\right)^{1/2}} \quad \text{and} \quad q = \frac{y}{\lambda\left(f^2 + x^2 + y^2\right)^{1/2}} \tag{18}$$

Substituting for p and q in equation (22) we can obtain an expression for $A(x,y)$ in terms of x

and y ; however, this result is somewhat complicated by the fact that p and q are each dependent on both x and y . When the above expressions are substituted for p and q , the optical Fourier transform is given by

$$F(p, q) = F\left(\frac{x}{\lambda\left(f^2 + x^2 + y^2\right)^{1/2}}, \frac{y}{\lambda\left(f^2 + x^2 + y^2\right)^{1/2}}\right)$$

$$= \iint A'\left(x_1, y_1\right) e^{-i2\pi\left[xx_1/\lambda\left(f^2+x^2+y^2\right)^{1/2}+yy_1/\lambda\left(f^2+x^2+y^2\right)^{1/2}\right]} dx_1\, dy_1 \tag{25}$$

A more desirable relationship would exist if p were directly proportional to x and if q were directly proportional to y . Then the light amplitude at a particular value of x would be related to a particular value of p and a similar relation would exist between q and y . As given by equation (18) p and x (also q and y) are not so simply related since p also depends on y (q also depends on x). In a later discussion the importance of a linear relation between the transform coordinates p,q and the spatial coordinates x, y will be shown.

To obtain a linear relation between p and x let us consider the series expansion of equation (18a).

$$p = \frac{x}{\lambda f}\left[1 - \frac{x^2 + y^2}{2f^2} + \frac{3}{8}\left(\frac{x^2 + y^2}{f^2}\right)^2 \cdots\right]$$

If we restrict our analysis to an area of the focal plane such that

$$\frac{x^2 + y^2}{2f^2} \ll 1$$

we can neglect all but the first term in the brackets to obtain the approximation

$$p = \frac{x}{\lambda f} \tag{26}$$

Similarly, we can obtain an approximation of q given by

$$q = \frac{y}{\lambda f} \tag{27}$$

Thus within a restricted area of the back focal plane the Fourier transform expression can be written as

$$F(p, q) = F\left(\frac{x}{\lambda f}, \frac{y}{\lambda f}\right) = \iint A'\left(x_1, y_1\right) e^{-i2\pi(xx_1/\lambda f + yy_1/\lambda f)} \, dx_1 \, dy_1 \tag{28}$$

The coordinates x and y in the back focal plane are then scaled representations of the frequencies p and q respectively. That is, light contributions corresponding to a spatial frequency p in the x_1 direction appear at the coordinate $x = \lambda fp$ in the back focal plane (similarly, contributions of spatial frequency q in y_1 direction appear at $y = \lambda fq$).

The actual restriction to be imposed on x and y for the above approximation will depend on how accurate a Fourier frequency value is required in a particular application. The error in the approximate frequency of equation (26) and (27) as a fraction of the exact value given by equations (18) is

$$E_f = \frac{\left(\frac{x}{\lambda f}\right)}{\left(\frac{x}{\lambda\left(f^2 + x^2 + y^2\right)^{1/2}}\right)} - 1 = \left(1 + \frac{x^2 + y^2}{f^2}\right)^{1/2} - 1 \tag{29}$$

We can let $r^2 = x^2 + y^2$ (r is the radius of a circle in the x, y plane) and express r in terms of multiples of the focal length f as given by

$$r = af \tag{30}$$

Substituting $r^2 = x^2 + y^2 = a^2 f^2$ in equation (29) we obtain

$$E_f = \left(1 + a^2\right)^{1/2} - 1 \tag{31}$$

The curve of Figure 8 gives the percent error (100 E_f) of the linear approximation of frequency as a function of a . For a less than 0.14 the error will be less than 1%. Thus the linear approximations

$$p = \frac{x}{\lambda f} \tag{26}$$

$$q = \frac{y}{\lambda f} \tag{27}$$

are accurate within 1% for values of x and y satisfying the restriction

$$\left(x^2 + y^2\right)^{1/2} = r \leqq 0.14f$$

For accuracies better than 1% smaller values of a must be imposed as given by equation (31) and Figure 8.

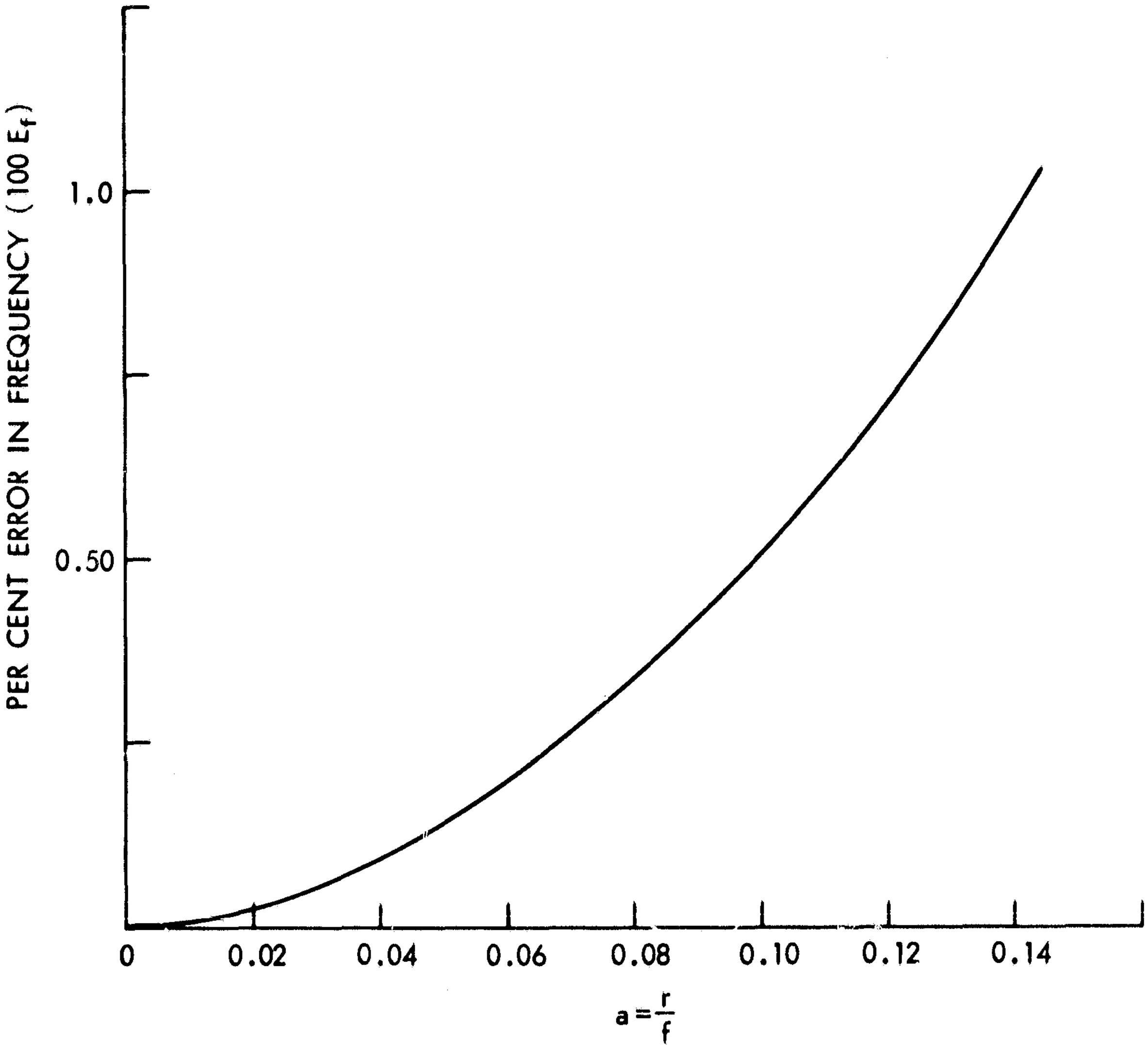


Figure 8—Percent error in linear frequency approximation.

Since our approximation requires that we limit our consideration to the area within a circle of radius $r_{max} = 0.14f$ (for accuracy within 1%), the maximum value of $x^2 + y^2$ is specified by:

$$\left(x^2 + y^2\right)_{max} = r_{max}^2 = .02f^2$$

Squaring the approximate expressions for p and q [equation (26) and (27)] and adding we obtain

$$p^2 + q^2 = \frac{x^2 + y^2}{f^2 \lambda^2} \tag{32}$$

Applying the restrictions on $x^2 + y^2$ to equation (32) we obtain

$$p^2 + q^2 \leq \frac{.02}{\lambda^2}$$

The maximum allowed value of p occurs when $q = 0$ and the maximum q occurs when $p = 0$:

$$p_{max} = \frac{.14}{\lambda} \quad \text{for} \quad q = 0 \text{ (i.e. } y = 0)$$

$$q_{max} = \frac{.14}{\lambda} \quad \text{for} \quad p = 0 \text{ (i.e. } x = 0)$$

As an example, consider green light of wavelength $\lambda = 5461 \times 10^{-8}$cm . In this case the simplified expressions for p and q are accurate within 1% for frequencies in the range given by:

$$\left(p^2 + q^2\right)^{1/2} \leq \frac{.14}{\lambda} = \frac{.14 \times 10^8}{5461} \approx 2560 \text{ cycles/cm.}$$

On the x axis $(y = 0, q = 0)$ the maximum frequency will be

$$p_{max} = 2560 \text{ cycles/cm.}$$

On the y axis $(x = 0, p = 0)$ the maximum frequency is likewise

$$q_{max} = 2560 \text{ cycle/cm.}$$

In practice the limitation of available techniques for controlling the input light distribution $A'(x_1, y_1)$ restrict maximum spatial frequencies to values below the 2560 cycles/centimeter restriction we have imposed above. Therefore, the linear approximations of the frequency components p and q [equation(26)and(27)] are applicable for practical systems and equation (22) and (23) can be expressed as:

$$A(x, y) = KF\left(\frac{x}{\lambda f}, \frac{y}{\lambda f}\right) \tag{32}$$

$$I(x, y) = \frac{\left|F\left(\frac{x}{\lambda f}, \frac{y}{\lambda f}\right)\right|^2}{\left[\lambda(f + d)\right]^2} \tag{33}$$

The accuracy of p and q used in equations (32) and (33) as given by (18) is determined by equation (31) as shown in Figure 8 for values of $a = \frac{r}{f}$. Thus for a given focal length f , the restriction on the maximum value of r determines the accuracy of the linear approximation introduced here. Of course, these equations also include the approximation assumed earlier in neglecting terms other than the $F(p, q)$ term. In the next section we will consider that approximation and we will determine if the restriction $x^2 + y^2 \leq 0.02f^2$ is also a sufficient restriction to assure the validity of neglecting terms other than $F(p,q)$.

APPROXIMATION LIMITS FOR THE FOURIER TRANSFORM REPRESENTATION

We will now return to the focused diffraction formula given by equation (20) as

$$A(x, y) = \frac{-ie^{ikR(x,y)}}{\lambda(f+d)} \iint \left[1 - \frac{1}{1 + \dfrac{f(f+d)}{x(x-x_1) + y(y-y_1)}} \right] A'(x_1, y_1)\, e^{-i2\pi(px_1+qy_1)}\, dx_1\, dy_1 \qquad (20)$$

and consider the limitations required to obtain the Fourier transform approximation given by equation (22). To obtain the form of a Fourier transform of $A'(x_1, y_1)$, the bracketed term must be approximated by a constant. This term can be assumed equal to one if we restrict the range of $x, y, x_1,$ and y_1 to satisfy the inequality

$$\frac{1}{1 + \dfrac{f(f+d)}{x^2 + y^2 - (xx_1 + yy_1)}} \ll 1$$

Referring back to equation (21) this approximation corresponds to making the second integral negligible compared to the first integral which has the form of a Fourier transform The complete term inside the brackets of equation (20) is effectively a weighting factor which varies the contribution from each point (x_1, y_1) to the point (x, y) . This factor represents the effect of the obliquity factor and path length attenuation. It is usually assumed that these effects are negligible and that the inequality condition is satisfied. In the following analysis we will attempt to present a more detailed quantitative discussion of this approximation.

Neglecting the variable term when it satisfies the inequality condition given above is an approximation of the light amplitude contribution from each point (x_1, y_1) to a point (x, y) . That is, the contribution $dA(x, y)$ at a point (x, y) from an infinitesimal region $dx_1\, dy_1$ about the point (x_1, y_1) is given exactly by

$$dA(x, y) = K \left[1 - \frac{1}{1 + \dfrac{f(f+d)}{x(x-x_1) + y(y-y_1)}} \right] A'(x_1, y_1)\, e^{-i2\pi(px_1+qy_1)}\, dx_1\, dy_1$$

and applying the approximation of neglecting the variable term inside the bracket we obtain

$$dA(x, y) = KA'(x_1, y_1)\, e^{-i2\pi(px_1+qy_1)}\, dx_1\, dy_1 \tag{34}$$

The total light amplitude $A(x, y)$ at a point (x, y) is obtained by integrating over the range of x_1 and y_1. The integration of equation (34) will yield an approximation for the total light amplitude $A(x, y)$ at least as accurate as the worse case of equation (34). That is, the greatest possible error would be given by the maximum value of the neglected term.

Thus to determine the limitations to be imposed, we will consider the maximum error introduced by neglecting the variable term to obtain equation (34). Denoting the error by the fraction E_A given by the ratio of the neglected term to the exact factor within the brackets of equation (20) we obtain:

$$E_A = \frac{x^2 + y^2 - (xx_1 + yy_1)}{f(f+d)} \tag{35}$$

To simplify our discussion we can express x, y, x_1 and y_1 in terms of polar coordinates r, ϕ, r_1, and ϕ_1. The relations between these coordinates are given by the equations

$$r^2 = x^2 + y^2 \qquad r_1^2 = x_1^2 + y_1^2$$

$$x = r\cos\phi \qquad x_1 = r_1 \cos\phi_1$$

$$y = r\sin\phi \qquad y_1 = r_1 \sin\phi_1$$

Substituting in equation (35) we obtain

$$E_A = \frac{r^2 - rr_1(\cos\phi\cos\phi_1 + \sin\phi\sin\phi_1)}{f(f+d)} \tag{36}$$

Using the identity $\cos(\phi - \phi_1) = \cos\phi\cos\phi_1 + \sin\phi\sin\phi_1$ equation (36) can be rewritten as

$$E_A = \frac{r^2 - rr_1\cos(\phi - \phi_1)}{f(f+d)} \tag{37}$$

The cosine term can take values between minus one and plus one. Since we are interested in the maximum error, we will consider the case for $\cos(\phi - \phi_1) = -1$. Equation (37) can be rewritten for cosine equal to minus one as

$$E_A = \frac{r^2 + rr_1}{f(f+d)} \tag{38}$$

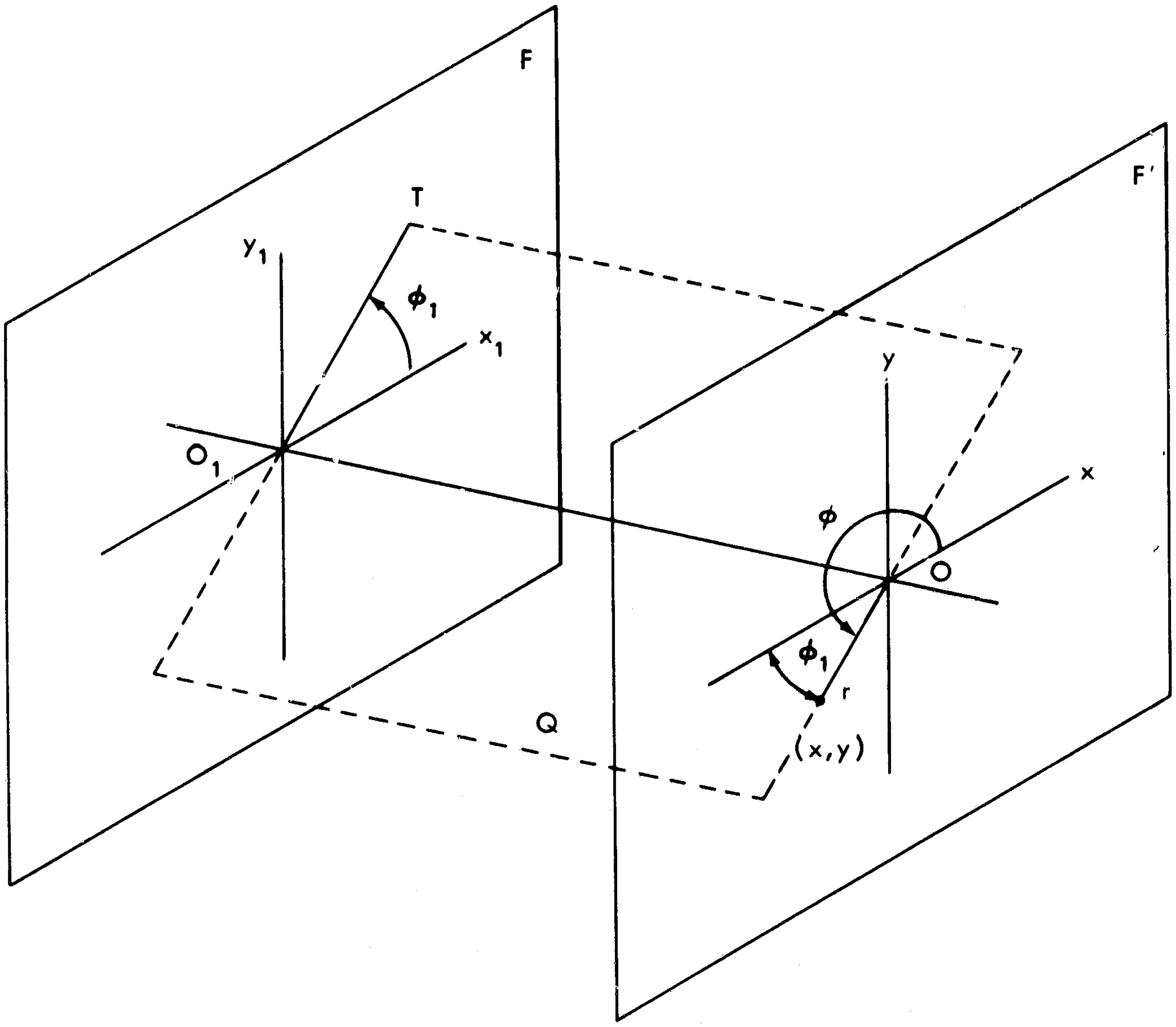


Figure 9—Diagram of conditions for maximum E_A.

Figure 9 shows the relative positions of points (x_1, y_1) and (x, y) for the case $\cos(\phi - \phi_1) = -1$. As shown by the figure, the maximum error defined by equation (38) applies to the light contributions from points (x_1, y_1) located on the line of intersection O_1T between the planes F and Q. The plane Q is a plane containing the optical axis and the point (x, y) in the back focal plane F'. The points (x_1, y_1) are further restricted to the portion of the line of intersection of F and Q which is on the side of the optical axis opposite from the point (x, y). From the geometry of the figure it is clear that the angle ϕ is equal to $\pi + \phi_1$. Thus $\phi - \phi_1$ is equal to π and cosine $(\phi - \phi_1) = \cos\pi = -1$ as required for the maximum E_A given by equation (38). For any point in the F plane which does not fall on the line O_1T the cosine term will be greater than -1 and the value of E_A will be less than that given by equation (38).

Examination of equation (38) shows that the error E_A increases as r and r_1 increase. Therefore, to determine the maximum value of E_A as a function of r and r_1 we need only to specify the maximum values of r and r_1 . Conversely, if we are interested in restricting the value of E_A to be less than or equal to a specified value the maximum values of r and r_1 must satisfy equation (38) for that particular value of E_A .

In order to analyze the relation between maximum E_A, r, and r_1 we must consider the interdependence of the maximum values of r and r_1 due to the limitations of a finite lens aperture. In our discussion, we will assume that diffraction effects at the end of the lens aperture are negligible.

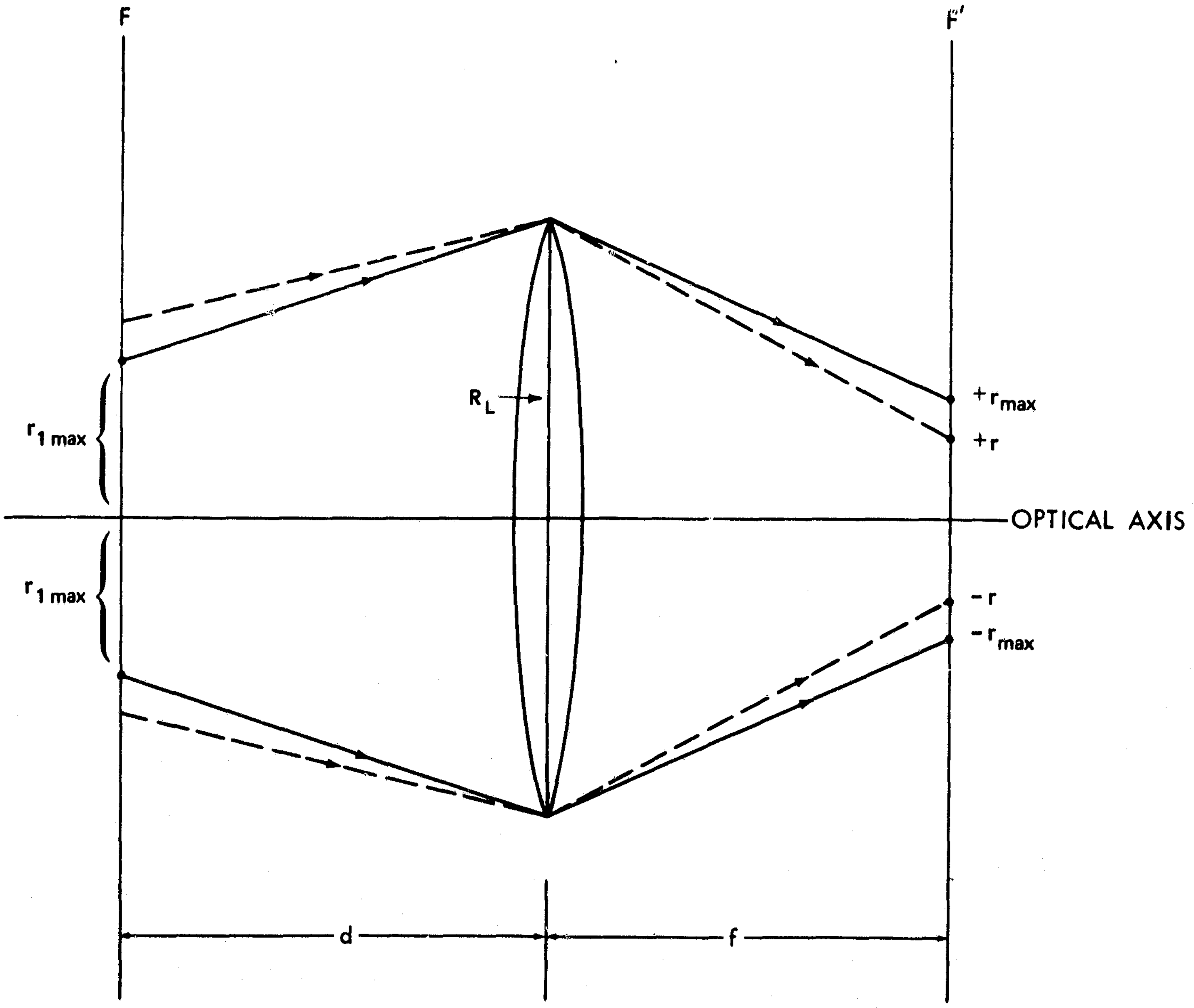


Figure 10—Lens aperture limitations on r_{max} and r_{1max}.

Figure 10 shows the extreme rays which can pass through a lens aperture of radius R_L to reach the points at the distance r_{max} from the optical axis. It should be apparent that any ray parallel to, but above the upper extreme ray; or parallel to, but below the lower extreme ray will be outside the lens aperture and will not pass through the lens. Thus any signal point outside the ray defined by $r_{l\,max}$ in Figure 10 cannot contribute to both of the points $+r_{max}$ and $-r_{max}$. For example, if the signal area extended upward beyond the $r_{l\,max}$ limit, the additional signal interval cannot contribute to the spectral point at $+r_{max}$ since the necessary light path will fall outside of the lens aperture. Under these circumstances the amplitude at the spectral point at $+r_{max}$ will not correspond to the entire signal but only to the interval below the $+r_{l\,max}$ limit. From this example it is apparent that the $r_{l\,max}$ limit given by Figure 10 defines the maximum signal interval over which every point contributes to the spectral points at $\pm r_{max}$.

The dashed lines in Figure 10 represent the extreme rays to a spectral point at a distance r which is less that r_{max}. It is seen that the extreme rays for such a case define a maximum signal interval longer than that obtained for r_{max}. This means that the signal interval defined by r_{lmax} increases as the spectral range of interest defined by r_{max} decreases. Thus we see that for a given maximum frequency (i.e. r_{max}) the maximum value of the signal interval $r_{l\,max}$ is limited by the lens aperture.

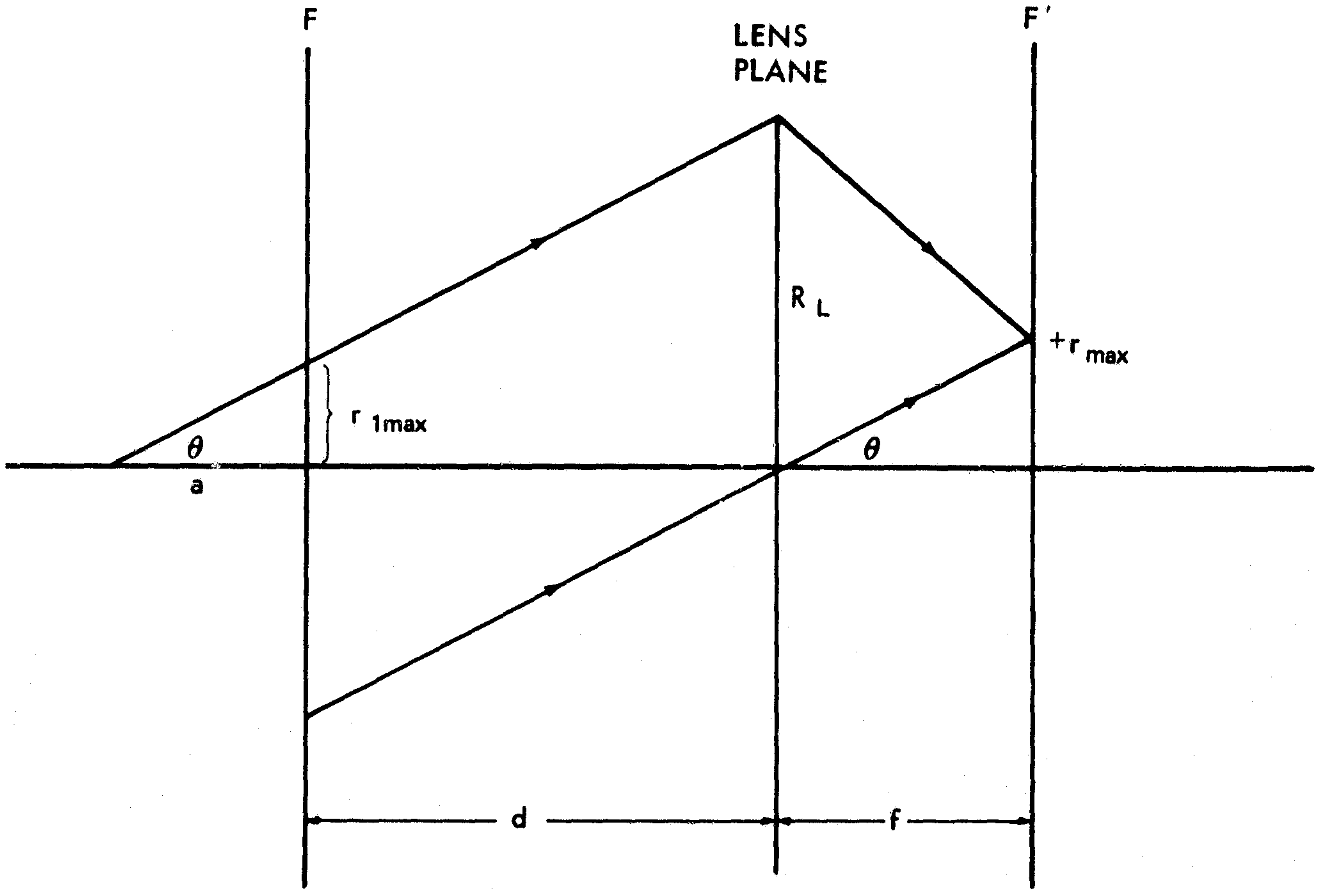


Figure 11—Geometry for relation between r_{max} and r_{1max}.

To derive an expression defining the relation between r_{max} and $r_{1\,max}$ we will use the geometry of Figure 11. This figure represents the upper extreme ray and the principal ray contributing to the spectral point at $+r_{max}$. Since the extreme ray must be parallel to the principal ray, the angles θ are equal and we can apply the principles of similar triangles to obtain

$$\frac{r_{max}}{f} = \frac{r_{1max}}{a} = \frac{R_L}{a + d}$$

From these relations we obtain two equations for a :

$$a = \frac{r_{1max}}{\left(\frac{r_{max}}{f}\right)} \quad \text{and} \quad a = \frac{R_L}{\left(\frac{r_{max}}{f}\right)} - d$$

Since the right hand sides of these equations must be equal, we obtain the result

$$r_{1max} = R_L - d\,\frac{r_{max}}{f} \qquad (39)$$

A lens is usually specified by its F stop which is defined as

$$F = \frac{f}{2R_L} \qquad (40)$$

Dividing both sides of equation (39) by f we obtain

$$\frac{r_{1max}}{f} = \frac{R_L}{f} - \frac{d}{f}\,\frac{r_{max}}{f} \qquad (41)$$

From equation (40) we find that $\frac{R_L}{f} = \frac{1}{2F}$ and substituting into equation (41) we can write

$$\frac{r_{1max}}{f} = \frac{1}{2F} - \frac{d}{f}\,\frac{r_{max}}{f} \qquad (42)$$

It is obvious from equation (42) as well as Figure 10 that $r_{1\,max}$ cannot be greater than the lens aperture radius R_L . Equation (42) defines the maximum allowed signal aperture radius $r_{1\,max}$ due to the limitations of the lens aperture. In practice the size of the signal aperture is specified by physical consideration or a desired size format. We can rearrange terms in equation (42) to define the maximum spectral term r_{max} as

$$\frac{r_{max}}{f} \;\; \frac{\frac{1}{2F} - \frac{r_{1max}}{f}}{\frac{d}{f}} \qquad (43)$$

Equation (43) defines the maximum allowable r for a given r_{1max} as determined by the restriction of a lens aperture. By rearranging terms in equation (38) we can obtain a second expression specifying the limitations on r_{max} required for an allowed error E_A .

$$\frac{r_{max}}{f} = \frac{1}{2}\frac{r_{1max}}{f}\left[\left(1 + \frac{4E_A\left(1+\frac{d}{f}\right)}{\left(\frac{r_1}{f}\right)^2}\right)^{1/2} - 1\right] \qquad (44)$$

To demonstrate the application of equation (43) and (44) we will consider the case for $\frac{r_{1max}}{f} = \frac{1}{5}$ (e.g. for $f = 100mm$, $r_1 = 20mm$). In Figure 12, we have plotted $\frac{r_{max}}{f}$ as a function of $\frac{d}{f}$ for the specified input aperture, $\frac{r_{1max}}{f} = \frac{1}{5}$. The curves labeled $F = 1.4$ and $F = 2$ correspond to equation (43) for the specified values of F . The curves labeled $E_A = 0.02$, $E_A = 0.01$, and $E_A = 0.005$ correspond to equation (44) for the specified values of E_A . The F curves specify the upper limit on $\frac{r}{f}$ due to the lens aperture and the E_A curves specify the upper limit for a given accuracy of the approximation. For a chosen value of $\frac{d}{f}$, the value of $\frac{r_{max}}{f}$ must be below both the F and E_A curve which apply to the particular system being considered.

For example, consider the case in which a lens with $F = 2$ is to be used and the maximum error to be allowed is $E_A = 0.02$. The greatest value allowed for $\frac{r}{f}$ corresponds to the point A which is the intersection of the $F = 2$ curve and the $E_A = 0.02$ curve. The value of $\frac{d}{f} = 0.5$ would be selected to obtain the value $\frac{r_{max}}{f} = 0.1$. For any other value of $\frac{d}{f}$ the limit on $\frac{r_{max}}{f}$ would be less than the maximum at point A . For $\frac{d}{f}$ less than 0. 5 the $E_A = 0.02$ curve specifies a tighter limit on $\frac{r_{max}}{f}$ while for $\frac{d}{f}$ greater than 0, 5 the $F = 2$ curve limits $\frac{r_{max}}{f}$. Of course any combination of $\frac{r_{max}}{f}$ and $\frac{d}{f}$ corresponding to a point below the curves is allowed; the curves only define the upper limit on $\frac{r_{max}}{f}$ for a given value of $\frac{d}{f}$.

The specification of a desired value of the maximum error E_A is not readily determined in practice. Since the error in the contribution from (x_1, y_1) varies from point to point, the total effect of the error can not be determined unless the integration of the exact expression given by equation (20) can be evaluated. To circumvent this difficulty we will consider a more or less logical selection of parameters and determine the maximum E_A specified by these parameters. The value of E_A determined will then specify the maximum error in the approximation we are considering.

In Figure 12, the dashed line corresponding to $\frac{r_{max}}{f} = 0.14$ represents the limit determined in the last sectio ' for an accuracy of better than 1% in the linear relation between spectral frequency and back focal plane coordinates. From our previous discussion of equation (38) we note that the error E_A decreases as $\frac{d}{f}$ increases. Therefore, if we do not wish to lower the previous limit of

$\frac{r_{max}}{f} = 0.14$, we can improve (decrease) the error E_A by using the largest possible value of $\frac{d}{f}$. Referring to Figure 12, we can note that for a lens with F = 2 , the maximum value of $\frac{d}{f}$ is 0.3 which is given by the intersection of the dashed line $\frac{r}{f} = 0.14$ and the F = 2 curve. As noted on Figure 12, the value of E_A at this point is 0.037 (from equation (38)). If we considered a lens with F = 1.4 , Figure 12 shows that we can increase $\frac{d}{f}$ to a value of 1.1 and reduce the error to E_A = 0.023. Thus we obtain the usual result that the lens of lower F provides the better characteristics (lower F implies larger lens aperture for given focal length). In addition to having the larger error E_A the F = 2 lens restriction of $\frac{d}{f} = 0.3$ presents practical problems—the lens mount and input aperture mount must be designed to allow for a small spacing (d = 3 cm for f = 10 cm) .

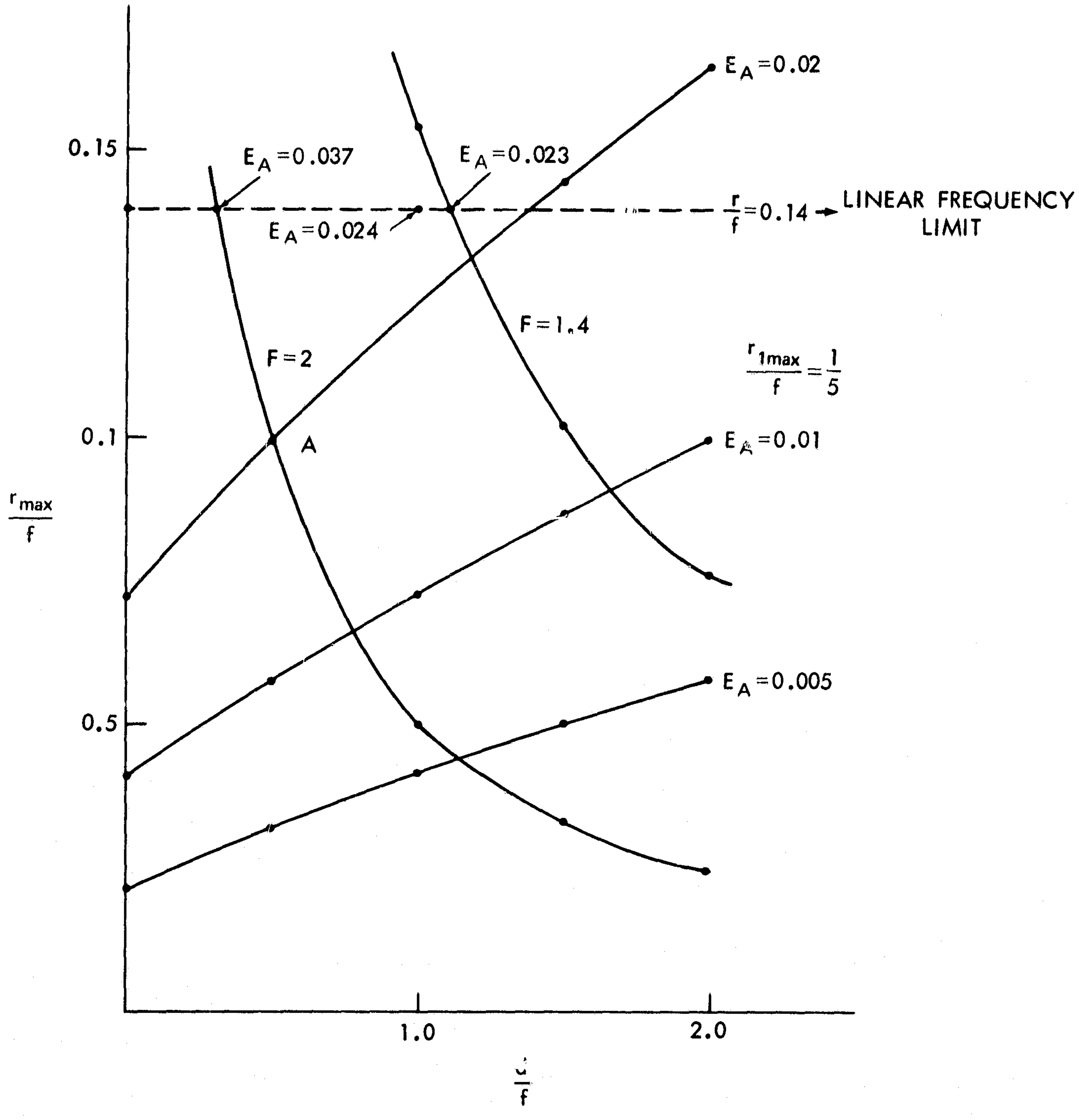


Figure 12—Limitation on maximum spectral term r_{max}/f.

In practice lower values of $\frac{r_{1max}}{f}$ and $\frac{r}{f}$ may be satisfactory. In such cases the error E_A would be less than the $E_A = 0.023$ determined here and higher F lenses (smaller lenses) may be used. Here we have considered an extreme case and determined what amounts to an extreme error $E_A = 0.023$ (or 2.3%). This extreme value of the error introduced by the Fourier transform approximation is quite reasonable and should be sufficient justification for using the transform approximation in most applications.

In the next sections we will consider the phase of the light distribution in the back focal plane F' . It will be shown that selecting $\frac{d}{f} = 1$ has advantages in reducing the phase factors not associated with the Fourier transform. In Figure 12 the point corresponding to $\frac{d}{f} = 1$ and $\frac{r_{max}}{f} =$ 0.14 is shown to have an error value $E_A = 0.024$. Thus it is seen that for the extreme case considered in the discussion above reducing the value of $\frac{d}{f}$ by one - tenth increases the error by 0.001. Such a slight increase in the error E_A is quite reasonable in terms of the advantage gained in the phase approximation treated in the next section. In addition, since the location of a plane at a value of $\frac{d}{f}$ can never be completely accurate, the displacement from the $F = 1.4$ curve allows a safety margin of + 10% allowable error in the location specified by $\frac{d}{f} = 1$ without exceeding the limitations imposed by the lens aperture.

Thus we have shown how equations (38), (43), and (44) can be used to determine and/or specify the parameter limits and the accuracy of the Fourier transform representation:

$$A(x, y) = \frac{-ie^{ikR(x,y)}}{\lambda(f+d)} \iint A'(x_1, y_1)\, e^{-i2\pi(px_1+qy_1)}\, dx_1\, dy_1 \tag{45}$$

In particular we have shown that for the maximum values $\frac{r_{max}}{f} = 0.14$ and $\frac{r_{1max}}{f} = \frac{1}{5}$, and the desirable choice of $\frac{d}{f} = 1$, the worse possible error in the amplitude values given by this equation is 2.4%. Since the term neglected in equation (20) is negative, the approximate amplitude given by equation (45) will be higher than the exact values by no more than 2.4%.

FOURIER TRANSFORM REPRESENTATION OF OPTICAL IMAGING

By limiting the area of consideration in the input plane

$$\left(\frac{r_{1max}}{f} < \frac{1}{5}\right)$$

and in the back focal plane

$$\left(\frac{r_{max}}{f} < .14\right),$$

we have shown that the light amplitude distribution $A(x, y)$ in the back focal plane of a lens is given with reasonable accuracy by

$$A(x, y) = \frac{-i e^{ikR(x,y)}}{\lambda(f + d)} F(p, q) \qquad (46)$$

where

$$p = \frac{x}{\lambda f} \qquad q = \frac{y}{\lambda f} \qquad (47)$$

$$R(x, y) = \frac{f^2 + df + x^2 + y^2}{\left(f^2 + x^2 + y^2\right)^{1/2}} = \frac{f^2 + df + r^2}{\left(f^2 + r^2\right)^{1/2}} \qquad (48)$$

$$F(p, q) = \iint A'\left(x_1, y_1\right) e^{-i2\pi(px_1 + qy_1)} \, dx_1 \, dy_1 \qquad (49)$$

As given by equation (49), $F(p, q)$ is the two-dimensional Fourier transform of the light amplitude listribution $A'(x_1, y_1)$ in a plane perpendicular to the optical axis and at a distance d in front of the lens.

As pointed out in the discussion of equations (22), (23) and (24), the phase term $e^{ikR(x,y)}$ is of concern only when a second lens is introduced to produce an image as shown in Figure 13.*

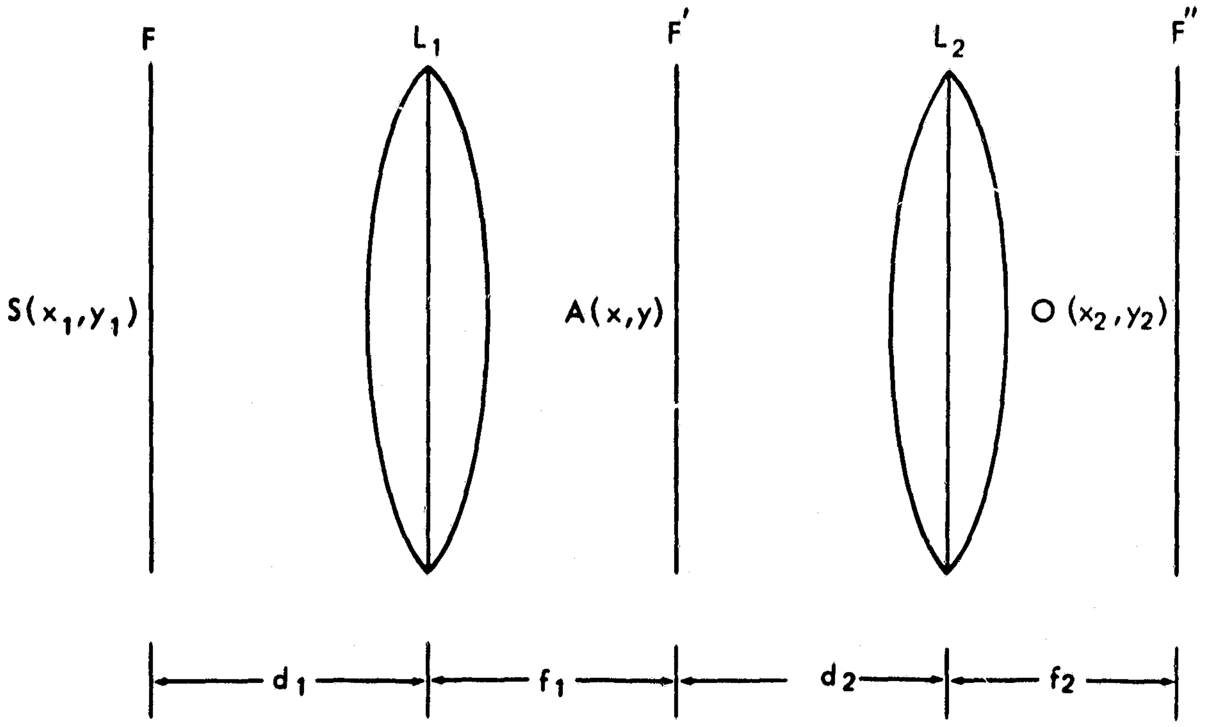


Figure 13—Two lens optical imaging system.

*We are considering only conventional optical systems here. If holographic techniques are considered, the phase factor in equation (46) would determine the form of the interference pattern produced by A(x, y) and a reference signal.

To simplify our notation we introduce K defined as

$$K = \frac{-i e^{ikR(x,y)}}{\lambda(f + d)}$$

and write equation (46) as

$$A(x, y) = KF(p, q)$$

In Figure 13, the light amplitude distribution $A'(x_1, y_1)$ in the input plane F is given as $S(x_1$
Using equation (49), (50), and (51), the light amplitude distribution $A(x, y)$ in the plane F'
focal plane of lens L_1) is given as

$$A(x, y) = K_1 \iint S(x_1, y_1)\, e^{-i2\pi(px_1+qy_1)}\, dx_1\, dy_1$$

where

$$p = \frac{x}{\lambda f_1} \qquad q = \frac{y}{\lambda f_1}$$

$$K_1 = \frac{-ie^{ikR_1(x,y)}}{\lambda\left(f_1 + d_1\right)}$$

$$R_1(x, y) = \frac{f_1^2 + d_1 f_1 + r^2}{\left(f_1^2 + r^2\right)^{1/2}}$$

Similarly, $A(x, y)$ is the input signal to the lens L_2 , and the light distribution $O(x_2, y_2)$ i
output plane F'' (back focal plane L_2) can be written as

$$O\left(x_2, y_2\right) = K_2 \iint A(x, y)\, e^{-i2\pi(p'x+q'y)}\, dx\, dy$$

where

$$p' = \frac{x_2}{\lambda f_2} \qquad q' = \frac{y_2}{\lambda f_2}$$

$$K_2 = \frac{-i e^{ikR_2(x_2, y_2)}}{\lambda (f_2 + d_2)} \tag{58}$$

$$R_2 (x_2, y_2) = \frac{f_2^2 + f_2 d_2 + r_2^2}{(f_2^2 + r_2^2)^{1/2}} \tag{59}$$

Substituting (52) into (56) we obtain an expression for the output image $O(x_2, y_2)$ in terms of the input image $S(x_1, y_1)$

$$O(x_2, y_2) = K_2 \iint e^{-i2\pi(p'x+q'y)} \, dx \, dy \left\{ K_1 \iint S(x_1, y_1) \, e^{-i2\pi(px_1+qy_1)} \, dx_1 \, dy_1 \right\} \tag{60}$$

We will assume that the function $S(x_1, y_1)$ allows the order of integration to be reversed and rewrite equation (60) as

$$O(x_2, y_2) = K_2 \iint S(x_1, y_1) \, dx_1 \, dy_1 \left\{ \iint K_1 \, e^{-i2\pi(px_1+qy_1+p'x+q'y)} \, dx \, dy \right\} \tag{61}$$

The integral within the brackets is complicated by the presence of the factor K_1 which contains an exponential dependent upon x and y. If we limited the values of x and y (i.e. r/f) so that the phase variation in K_1 can be considered negligible, the K_1 factor can be taken outside the integrals giving

$$O(x_2, y_2) = K_2 K_1 \iint S(x_1, y_1) \, dx_1 \, dy_1 \left\{ \iint e^{-i2\pi(px_1+qy_1+p'x+q'y)} \, dx \, dy \right\} \tag{62}$$

We now consider the integral within the brackets and substitute for p, q, p' and q' from equations (53) and (57)

$$\iint e^{-i2\pi(px_1+qy_1+p'x+q'y)} \, dx \, dy = \int e^{-i2\pi(x_1/\lambda f_1 + x_2/\lambda f_2)x} \, dx \int e^{-i2\pi(y_1/\lambda f_1 + y_2/\lambda f_2)y} \, dx \tag{63}$$

Up to this point we have not mentioned the limits of integration. Due to the presence of aperture in an optical system the signals exist only over a finite range of the aperture coordinates. However, since the signals are zero outside this range (e.g. $S(x_1, y_1) = 0$ outside the aperture in the F plane) the contribution to the integral beyond the aperture limits will also be zero. Thus, we can take the limits of integration to be from $-\infty$ to $+\infty$. These limits are in agreement with the Fourier transform integrals.

The Dirac delta function can be defined by the integral equation:

$$\delta(x - a) = \int_{-\infty}^{\infty} e^{-i2\pi\mu(x-a)}\, d\mu \tag{64}$$

Comparing each of the integrals on the right side of (63) with the integral in equation (64) we find

$$\begin{aligned}\iint e^{-i2\pi(px_1+qy_1+p'x+q'y)}\, dx\, dy &= \delta\left(\frac{x_1}{\lambda f_1} + \frac{x_2}{\lambda f_2}\right)\delta\left(\frac{y_1}{\lambda f_1} + \frac{y_2}{\lambda f_2}\right) \\ &= \lambda^2 f_1^2\, \delta\left(x_1 + \frac{f_1}{f_2}\, x_2\right)\delta\left(y_1 + \frac{f_1}{f_2}\, y_2\right)\end{aligned} \tag{65}$$

In the last step of equation (65) we have used the identity

$$\delta(ax) = \frac{1}{|a|}\, \delta(x)$$

Equation (65) is valid only if x and y range from $-\infty$ to $+\infty$. In optical systems this is not the case since the range of the coordinates x and y is limited as demonstrated in our previous discussion. However, we assume (65) valid to simplify our discussion. Substituting equation (65) into equation (62) we obtain

$$O(x_2, y_2) = K_2 K_1 \lambda^2 f_1^2 \iint S(x_1, y_1)\, \delta\left(x_1 + \frac{f_1}{f_2}\, x_2\right)\delta\left(y_1 + \frac{f_1}{f_2}\, y_2\right) dx_1\, dy_1 \tag{66}$$

Now, we can make use of the sifting property of the Dirac delta function which is defined by

$$\int F(x)\, \delta(x + a)\, dx = F(-a)$$

Applying this property of the delta function to equation (66) we obtain

$$O(x_2, y_2) = K_2 K_1 \lambda^2 f_1^2\, S\left(-\frac{f_1}{f_2}\, x_2, -\frac{f_1}{f_2}\, y_2\right) \tag{67}$$

In deriving equation (67) we assumed that K_1 was approximately constant. The factor K_2 is variable only in phase as seen by referring to equation (58). Since only intensity is seen or measured, the phase variations of K_2 can be ignored and equation (67) can be interpreted as giving the output image $O(x_2, y_2)$ in terms of a proportionality factor $(K_2\, K_1 \lambda^2\, f_1^2)$ multiplying the original input signal S expressed in the new coordinates

$$\left(-\frac{f_1}{f_2}x_2, -\frac{f_1}{f_2}y_2\right)$$

To clarify the significance of these new coordinates, let us consider the relation between

$$S(x_1, y_1) \rightarrow S\left(-\frac{f_1}{f_2}x_2, -\frac{f_1}{f_2}y_2\right) \tag{68}$$

Since the two sides of relation (68) correspond point for point, we find that (x_1, y_1) and (x_2, y_2) coordinates are related by

$$x_1 = -\frac{f_1}{f_2}x_2 \qquad y_1 = -\frac{f_1}{f_2}y_2 \tag{69}$$

Equation (69) represents the fact that a point of the signal which was originally at the coordinates x_1 and y_1 will be imaged to the point at

$$x_2 = -\frac{f_2}{f_1}x_1 \quad \text{and} \quad y_2 = -\frac{f_2}{f_1}y_1 \ .$$

The magnification in an optical image is defined as the ratio of the imaged coordinate of a point to the original coordinate

$$m_x = \frac{x_2}{x_1} = -\frac{f_2}{f_1}$$

$$m_y = \frac{y_2}{y_1} = -\frac{f_2}{f_1} \tag{70}$$

Equations (70) were written separately although it is apparent that here the magnification is the same in any direction. In some cases it is possible to obtain different magnifications in different directions (e.g. cylindrical lens system). Equations (67) and (70) show that the output image is proportional to the input image with a change in scale. The minus sign which appears in equation (67) and (70) represents an inversion of the image.

In many applications, there is no requirement for a magnified image. In such cases, we could use lenses of equal focal length $f_1 = f_2$ and obtain a magnification $m = -1$.

For the case $f_1 = f_2 = f$, equation (67) becomes

$$O(x_2, y_2) = K_2 K_1 \lambda^2 f^2 S(-x_2, -y_2) \tag{71}$$

Thus, for equal focal length lenses the output image $O(x_2, y_2)$ is proportional to an inverted replica of the input signal.

It is for this case $(f_1 = f_2 = f)$, that the optical imaging process can be described as consecutive Fourier transforms. This can be shown by replacing f_2 by f in equation (57) and substituting for p' and q' in equation (56) to obtain

$$O(x_2, y_2) = K_2 \iint A(x, y)\, e^{-i2\pi(x_2 x/\lambda f + y_2 y/\lambda f)}\, dx\, dy \qquad (72)$$

by replacing f_1 by f in equation (53) we find

$$p = \frac{x}{\lambda f}, \qquad q = \frac{y}{\lambda f}, \qquad dx = \lambda f\, dp, \qquad dy = \lambda f\, dq \qquad (73)$$

Substituting (73) into equation (72) we obtain the result

$$O(x_2, y_2) = K_2 \lambda^2 f^2 \iint A(x, y)\, e^{-i2\pi(px_2 + qy_2)}\, dp\, dq \qquad (74)$$

Using equations (51), (52), (71) and (74) we can express the two step process of optical imaging as

$$A(x, y) = K_1 F(p, q) = K_1 \iint S(x_1, y_1)\, e^{-i2\pi(px_1 + qy_1)}\, dx_1\, dy_1 \qquad (75)$$

$$O(x_2, y_2) = K_2 K_1 \lambda^2 f^2 S(-x_2, -y_2) = K_2 K_1 \lambda^2 f^2 \iint F(p, q)\, e^{-i2\pi(px_2 + qy_2)}\, dp\, dq \qquad (76)$$

where

$$p = \frac{x}{\lambda f}, \qquad q = \frac{y}{\lambda f}$$

$$K_1 = \frac{-ie^{ikR_1(x,y)}}{\lambda(f + d_1)} \qquad K_2 = \frac{-ie^{ikR_2(x_2,y_2)}}{\lambda(f + d_2)}$$

$$R_1(x, y) = \frac{f^2 + fd_1 + r^2}{(f^2 + r^2)^{1/2}} \qquad R_2(x_2, y_2) = \frac{f^2 + fd_2 + r_2^2}{(f^2 + r_2^2)^{1/2}}$$

$$r^2 = x^2 + y^2 \qquad r_2^2 = x_2^2 + y_2^2$$

The last expression in equation (76) assumes that the factor K_1 can be considered constant in phase over the range of the values p and q (i.e. x and y). This approximation is the subject we are about to consider and here we are showing the advantages of the resulting expression. To appreciate the significance of equations (75) and (76) let us consider the standard Fourier transform equations using our notation

$$F(p, q) = \iint S(x_1, y_1)\, e^{-i2\pi(px_1+qy_1)}\, dx_1\, dy_1 \tag{77}$$

$$S(x_1, y_1) = \iint F(p, q)\, e^{+i2\pi(px_1+qy_1)}\, dp\, dq \tag{78}$$

$$S(-x_1, -y_1) = \iint F(p, q)\, e^{-i2\pi(px_1+qy_1)}\, dp\, dq \tag{79}$$

Equation (77) is usually referred to as the Fourier transform while equation (78) is the inverse Fourier transform. Note that the exponent of equation (78) is positive and that of equation (79) is negative. Since the optical transform produced by a lens has a negative exponent, the inverse transform defined by equation (78) never appears in optical systems. The second lens in an optical system such as that of Figure 13 produces a Fourier transform of a Fourier transform as represented by equation (79). Note that the inversion (or change of sign of the coordinate) is introduced by the second Fourier transform whereas an inverse transform would not invert the signal. Thus, comparing equations (75) and (76) with equation (77) and (79), we note that the optical imaging process of two lenses is described by two successive Fourier transforms relations. Except for determining the absolute amplitudes involved, the constants in front of the integrals of equations (75) and (76) do not affect the form of the variations. In most cases only the relative amplitudes are of interest and the constants are dropped.

The advantage of the Fourier transform representation described by equation (75) and (76) can be shown by considering the introduction of a filter in the F' plane. If we know the transmission characteristics of the filter, we can determine a function $M(p,q)$ which represents the fraction of incident light amplitude passed at each coordinate corresponding to the values of p and q. The filtered output $O_f\,(x_2, y_2)$ is then given by equation (76) if we replace $F(p,q)$ by $M(p,q)\,F(p,q)$

$$O_f\,(x_2, y_2) = K_2 K_1 \lambda^2 f' \iint M(p, q)\, F(p, q)\, e^{-i2\pi(px_2+qy_2)}\, dp\, dq \tag{80}$$

Thus, the specification of a filter for a particular application can be determined uniquely when the Fourier transform representation is used.

We might note at this point that the Fourier transform representation of equations (75) and (76) require the use of lenses of equal focal length. The specification of a filter for the case of unequal focal lengths is exactly the same; however, the Fourier transform relation of equation (76) is modified by introducing a factor of f_1/f_2 in the exponent to account for the magnification.

This additional f_1/f_2 results in a magnified filtered image which contains the same information as the filtered image $O_f(x_2, y_2)$ given by equation (80). The only difference (neglecting the phase of K_2) is in the scale. Throughout the remaining part of this report we will consider the special case of equal focal length lenses to simplify our analysis.

ELIMINATION OF UNDESIRABLE PHASE VARIATIONS

Now that we have seen the significance of the Fourier transform representation of optical imaging, we will consider the approximation involved in the derivation of equation (76). The actual relation corresponding to equation (76) can be written as

$$O(x_2, y_2) = K_2 \lambda^2 f^2 \iint K_1 F(p, q)\, e^{-i2\pi(px_2 + qy_2)}\, dp\, dq \tag{81}$$

The term K_1 appearing in the integral was defined as

$$K_1 = \frac{-ie^{ikR_1(x,y)}}{\lambda(f + d_1)} \tag{82}$$

where

$$R_1(x, y) = \frac{f^2 + fd_1 + r^2}{(f^2 + r^2)^{1/2}} \qquad (r^2 = x^2 + y^2) \tag{83}$$

Comparing equations (81) and (76) it is apparent that equation (76) is valid only if the phase variation of K_1 can be neglected over the range of the integration variables p and q (or x and y). We will now proceed to analyze this requirement.

For reasons that will become evident, we would like to express $R_1(x, y)$ in the form of:

$$R_1(x, y) = (f + d_1) + P(f - d_1) + Qf \tag{84}$$

$$= f(1 + P + Q) + d_1(1 - P)$$

Equation (83) can be rewritten as

$$R_1(x, y) = \frac{f^2 + r^2}{(f^2 + r^2)^{1/2}} + \frac{d_1 f}{(f^2 + r^2)^{1/2}} = f\left(1 + \frac{r^2}{f^2}\right)^{1/2} + d_1\left(1 + \frac{r^2}{f^2}\right)^{-1/2} \tag{85}$$

Equating the coefficients of f and d_1 in the final forms of equations (84) and (85) we obtain

$$1 - P = \left(1 + \frac{r^2}{f^2}\right)^{-1/2}$$

$$1 + P + Q = \left(1 + \frac{r^2}{f^2}\right)^{1/2}$$

Solving for P and Q we obtain

$$P = 1 - \left(1 + \frac{r^2}{f^2}\right)^{-1/2}$$

$$Q = \left(1 + \frac{r^2}{f^2}\right)^{1/2} + \left(1 + \frac{r^2}{f^2}\right)^{-1/2} - 2 = 2\left[\frac{1 + \frac{r^2}{2f^2}}{\left(1 + \frac{r^2}{f^2}\right)^{1/2}} - 1\right]$$

Substituting for P and Q, equation (84) can be written

$$R_1(x, y) = \left(f + d_1\right) + \left(f - d_1\right)\left[1 - \frac{1}{\left(1 + \frac{r^2}{f^2}\right)^{1/2}}\right] + 2f\left[\frac{1 + \frac{r^2}{2f^2}}{\left(1 + \frac{r^2}{f^2}\right)^{1/2}} - 1\right] \tag{86}$$

The K_1 term given by equation (82) can be rewritten using equation (86)

$$K_1 = \left[\frac{-ie^{ik(f+d_1)}}{\lambda\left(f - d_1\right)}\right] e^{ik(f-d_1)[1-(1+r^2/f^2)^{-1/2}]} \; e^{i2kf[(1+r^2/2f^2)(1+r^2/f^2)^{-1/2}-1]} \tag{87}$$

The terms grouped within the first brackets of equation (87) are constant and therefore can be taken out from under the integral sign in equation (81). The remaining exponentials in equation (87) are phase factors which depend on the variables of integration. The exponential of the remaining terms must be limited so that the phase variations can be considered negligible.

Since we are going to consider restrictions on the value of $\frac{r}{f}$ so that the phase terms can be considered negligible, we can simplify the exponentials of equation (87) by expanding in power series of $\left(\frac{r^2}{f^2}\right)$ and drop all but the first terms of the expansions. Expanding the exponent of the first exponential we obtain

$$e^{ik(f-d_1)[1-(1+r^2/f^2)^{-1/2}]} = e^{ik(f-d_1)\left[1-\{1-(r^2/2f^2)+3/8(r^2/f^2)^2\cdots\}\right]}$$

$$\simeq e^{ik(f-d_1)(r/f)^2/2}$$

Similarly, we expand the second exponential and obtain

$$e^{i2kf[(1+r^2/2f^2)(1+r^2/f^2)^{-1/2}-1]} = e^{i2kf\left[-1+(1+r^2/2f^2)\{1-(r^2/2f^2)+3/8(r^2/f^2)^2\cdots\}\right]}$$

$$\simeq e^{ikf(r/f)^4/4}$$

Substituting these approximate terms in equation (87) we obtain

$$K_1 = \frac{-ie^{ik(f+d_1)}}{\lambda(f+d_1)} e^{ik(f-d_1)(r/f)^2/2} e^{ikf(r/f)^4/4} \tag{88}$$

For values of $\left(\frac{r}{f}\right)$ less than our previous limit of 0.14, the approximations in each of the phase terms is accurate within 2% of its exact values. It shall be noted that neglecting terms in the exponentials as we have done here is valid only since we are going to consider phase variations less than one cycle.

Since we are trying to eliminate the phase variations of K_1, equation (88) indicates that the optimum choice of the distance d_1 equal to f eliminates the first phase term. Thus for the case when d_1 is chosen equal to f, equation (88) can be reduced to

$$K_1 = \left[\frac{-ie^{i2kf}}{2\lambda f}\right] e^{ikf(r/f)^4/4} \qquad \left(\text{for } d_1 = f\right) \tag{89}$$

If we substitute equation (89) for K_1 in equation (81) we obtain

$$O(x_2, y_2) = \frac{-iK_2\lambda f e^{i2kf}}{2} \iint e^{ikf(r/f)^4/4} F(p, q) e^{-i2\pi(px_2+qy_2)} dp\,dq \tag{90}$$

When the value of $\left(\frac{r}{f}\right)$ is limited so that the phase factor appearing under the integral of equation (90) can be neglected, the output image $O(x_2, y_2)$ is given by the equation

$$O(x_2, y_2) = K_1 K_2 \lambda^2 f^2 \iint F(p, q) e^{-i2\pi(px_2+qy_2)} dp\,dq \tag{91}$$

where

$$K_1 = \frac{-i e^{i2kf}}{2\lambda f}$$

Equation (91) is identical to equation (76) which we have shown to be the desired form for the Fourier transform representation of optical imaging.

To derive a specification for the maximum limit on $\left(\frac{r}{f}\right)$ which allows the variable term in K_1 to be neglected, we will consider the effect of the phase term for a particular $F(p, q)$.

$$F(p, q) = \delta(q)\left[A_0\,\delta(p) + \frac{B}{2}\left\{\delta(p - p_0) + \delta(p + p_0)\right\}\right] \qquad (92)$$

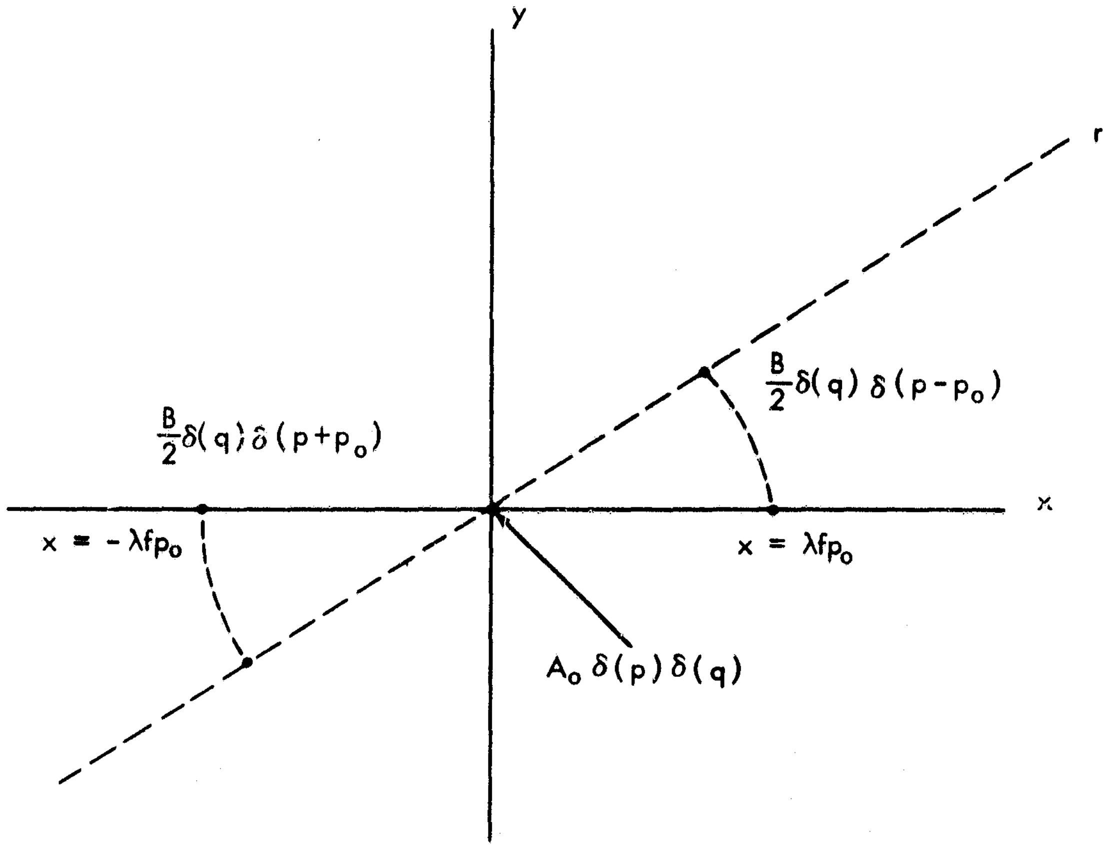

y
r
x
$\frac{B}{2}\delta(q)\,\delta(p-p_o)$
$\frac{B}{2}\delta(q)\,\delta(p+p_o)$
$x = -\lambda f p_o$
$x = \lambda f p_o$
$A_o\,\delta(p)\,\delta(q)$

Figure 14—Location of frequency terms in the spectrum plane.

The locations of the frequency terms contained in equation (92) are diagrammed in Figure 14. Since, by definition, the delta function $\delta(q)$ is equal to zero for q unequal to zero, equation (92) represents the spectrum of a signal which varies only in one - dimension. That is, there are no frequency components in the y direction; therefore, the signal is constant with respect to the y coordinate. Rather than interpret equation (92) as the spectrum of a particular signal, we can also assume that we are considering only three sample points of a more general spectrum. Since there is nothing to single out the x direction in an optical system, our analysis will apply to a set of spectral points along any radial axis in the frequency plane as indicated by the r axis in Figure 14. This is obvious if we consider the fact that we can arbitrarily select any orientation for our x, y coordinate axes. It can also be shown that the terms which we will use to specify a maximum limit on $\left(\frac{r}{f}\right)$ also apply to the general case. We will, therefore, simply interpret the results for the special case of equation (92) as a general criteria for neglecting the undesired phase factor in equation (90).

Substituting for $F(p, q)$ as given by equation (92) and applying the interpretation of our discussion above we can simplify equations (90) and (91) to the forms:

$$\underline{O(x_2)} = K\int e^{ikf(x/f)^4/4}\left[A_0\,\delta(p) + \frac{B}{2}\left\{\delta(p-p_0) + \delta(p+p_0)\right\}\right] e^{-i2\pi p x_2}\,dp \tag{93}$$

$$O(x_2) = K\int\left[A_0\,\delta(p) + \frac{B}{2}\left\{\delta(p-p_0) + \delta(p+p_0)\right\}\right] e^{-i2\pi p x_2}\,dp \tag{94}$$

The new factor K in equation (93) and (94) is defined as

$$K = \frac{-iK_2\,\lambda\, f\, e^{i2kf}}{2} \tag{95}$$

The integral with respect to q was taken by applying the sifting property of the delta function $\delta(q)$. The y_2 dependence has been dropped on the left side of equations (93) and (94) since the output image varies only with respect to the x_2 coordinate. The left side of equation (93) is underlined so that we can identify the ensuing results of equation (93) and (94) throughout the remainder of our discussion.

The first exponential in equation (93) can be rewritten using the definition $p = \frac{x}{\lambda f}$ and the integration with respect to p is performed simply by applying the sifting property of the delta function:

$$\underline{O(x_2)} = K\int e^{ikf(\lambda p)^4/4}\left[A_0\,\delta(p) + \frac{B}{2}\left\{\delta(p+p_0) + \delta(p-p_0)\right\}\right] e^{-i2\pi p x_2}\,dp$$

$$= K\left[A_0 + \frac{B}{2}\, e^{ikf(\lambda p_0)^4/4}\left\{e^{i2\pi p_0 x_2} + e^{-i2\pi p_0 x_2}\right\}\right]$$

$$\underline{O(x_2)} = K\left[A_0 + B\, e^{ikf(\lambda p_0)^4/4}\cos 2\pi p_0\, x_2\right] \tag{96}$$

The result after integrating equation (94) is similar to equation (96) except for the exponential which appears in equation (96):

$$O(x_2) = K\left[A_0 + B\cos 2\pi p_0 x_2\right] \tag{97}$$

Equations (96) and (97) represent the amplitude distribution of the output image produced by the frequency plane distribution $F(p, q)$ given by equation (92). Equation (96) represent the output $\underline{O(x_2)}$ when the phase factor is considered and equation (97) represent the output $O(x_2)$ when the phase factor is neglected. Comparing equation (96) and (97), a criteria for neglecting the phase factor is still not very apparent since the significance of the exponential is not very clear.

If we consider the observation of the output image, we must deal with the intensity rather than the amplitude as given by equations (96) and (97). The intensities are given by the relations

$$\underline{I(x_2)} = \underline{O(x_2)\,O^*(x_2)} = \underline{\left|O(x_2)\right|}^2 \tag{98}$$

$$I(x_2) = O(x_2)\,O^*(x_2) = \left|O(x_2)\right|^2 \tag{99}$$

The starred terms in equation (98) and (99) represent the complex conjugate of the unstarred terms. Using equations (96) and (97) in equations (98) and (99) respectively, we obtain

$$\underline{I(x_2)} = |K|^2\left[A_0^2 + B^2\cos^2 2\pi p_0 x_2 + 2A_0 B\cos\frac{kf}{4}(\lambda p_0)^4 \cos 2\pi p_0 x_2\right] \tag{100}$$

$$I(x_2) = |K|^2\left[A_0^2 + B^2\cos^2 2\pi p_0 x_2 + 2A_0 B\cos 2\pi p_0 x_2\right] \tag{101}$$

Now comparing equation (100) and (101), we find that the phase factor introduces a cosine factor which attenuates the $\cos 2\pi p_0 x_2$ component in the observed image. In the general case, the amplitude B of any one component will be considerably less than the component A_0. Therefore, the third term in equations (100) and (101) represent the larger of the two x_2 dependent terms in the image intensity. It is desirable to limit the maximum value of p_0 to obtain a value of $\cos\frac{kf}{4}(\lambda p_0)^4$ as near to one as possible so that the Fourier transform representation used in deriving equation (101) can be considered a good approximation.

We can express the cosine term as a function of x by the definition $p = \frac{x}{\lambda f}$ and since our results will apply to the general case we can replace x by the more general notation r. Thus we can express the cosine term as

$$\cos \frac{kf}{4} (\lambda p_0)^4 = \cos \frac{kf}{4} \left(\frac{r}{f}\right)^4 \qquad (102)$$

In equation (102) the frequency p_0 is given the general interpretation of a spatial frequency in the direction of an r axis (see Figure 14) and p_0 and r are related by $p_0 = \frac{r}{\lambda f}$. That is, equation (102) applies to the general case of a spectrum along any radial axis r in the back focal plane F'. We can express $\left(\frac{r}{f}\right)$ as a multiple of $\left(\frac{4}{kf}\right)^{1/4}$ by defining a factor m by the relation

$$\frac{r}{f} = m\left(\frac{4}{kf}\right)^{1/4} \qquad (103)$$

Substituting for $\frac{r}{f}$ in equation (102) we obtain

$$\cos \frac{kf}{4} \left(\frac{r}{f}\right)^4 = \cos m^4 \qquad (104)$$

We can recognize the ultimate limit on m by noting that for $m = \left(\frac{\pi}{2}\right)^{1/4}$ the cosine term as given by equation (104) is zero (i.e. $\cos \frac{\pi}{2} = 0$). For this value of m the Fourier transform result of equation (101) is completely in error with respect to the third term, since the cosine term present in equation (100) is zero and the third term is eliminated. Thus for $m = \left(\frac{\pi}{2}\right)^{1/4}$ our ideal two step Fourier transform representation yields a term which does not exist in the actual image given by equation (100)*. For values of m less than $\left(\frac{\pi}{2}\right)^{1/4}$ the cosine of equation (104) has non-zero values as shown by the curve of Figure 15. Selecting a limit on m is based on specifying how accurately the third term of equations (100) and (101) should agree. A value of $m = 0$ is necessary to have complete agreement between equation (100) and (101); however, $m = 0$ corresponds to a frequency $p_0 = 0$ which corresponds to a dc term only. Thus a compromise limit must be established between the limits $m = 0$ and $m = \left(\frac{\pi}{2}\right)^{1/4}$.

To determine the limitation on m, it is necessary to specify the desired accuracy of the third term in equation (101) as compared to the third term in equation (100). Again, the accuracy of our approximation can be given in terms of a fractional error E_ϕ defined as

$$E_\phi = \frac{1 - \cos m^4}{\cos m^4} = \frac{1}{\cos m^4} - 1 \qquad (105)$$

*This accounts for the frequent neglecting of phase terms only when less than $\pi/2$ which appears in many references dealing with approximate solutions of diffraction problems.

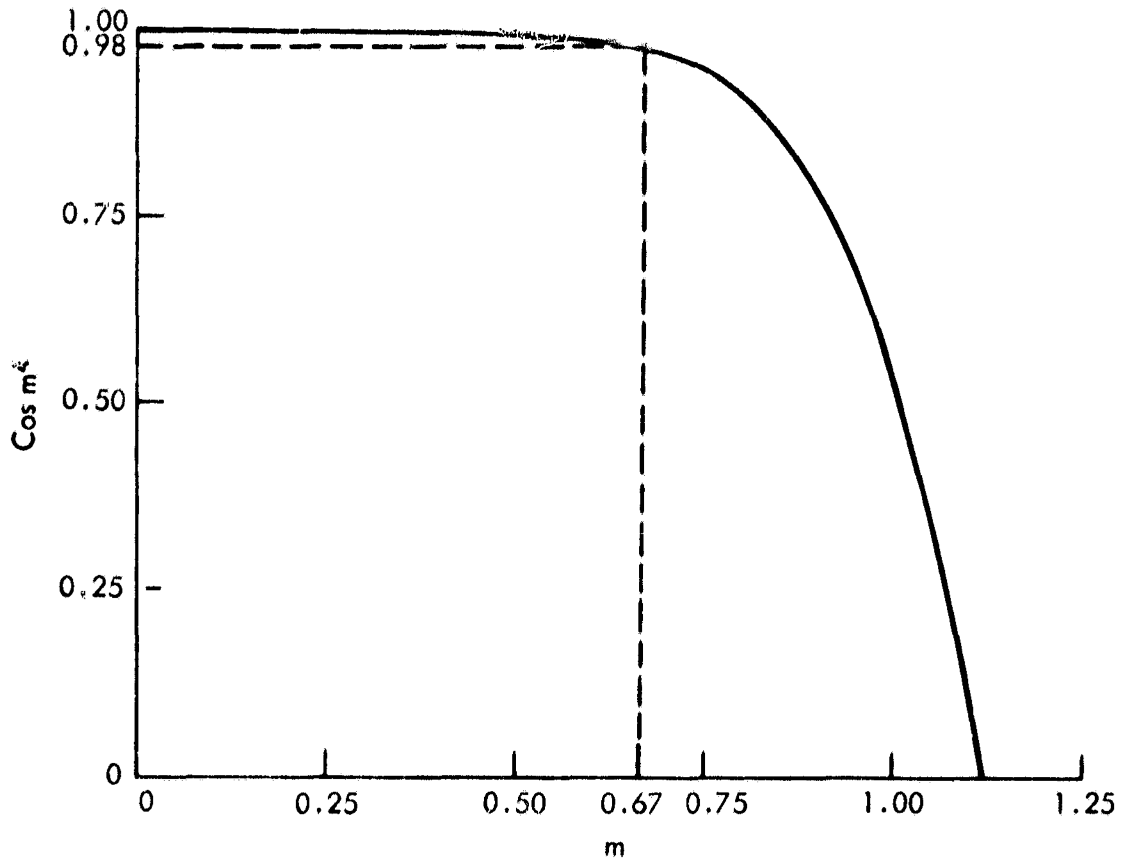


Figure 15—Graph of cos m^4 vs. m.

The error in the third term of equation (101) is then $+100\,E_\phi\%$ compared to the exact term in equation (100). Note that the error E_ϕ does not represent a fraction of the total image intensity. The error E_ϕ corresponds only to a particular term in the image intensity. In the general case there would be a series of such terms and the maximum E_ϕ would be determined by equation (106). This E_ϕ would represent the maximum error in terms of the form of the third term in equation (101) and would correspond to the term involving the highest frequency of interest.

Specifying the maximum allowable error is rather arbitrary and will usually depend on the particular application considered. However, for an example we can consider specifying a limit of 2% accuracy for our approximation, i.e. $E_\phi = 0.02$. Substituting in equation (105) we obtain the condition for the maximum value of m:

$$\cos m^4 \geq \frac{1}{1.02} = 0.98 \qquad (106)$$

From Figure 15 we find that the relation (106) requires values of m less than or equal to 0.67. Using the maximum value $m = 0.67$ equation (103) becomes

$$\left(\frac{r}{f}\right)_{max} = 0.67\left(\frac{4}{kf}\right)^{1/4} \qquad (107)$$

To obtain a numerical result for comparison with our previous limit $\frac{r}{f}_{max} = 0.14$, we will again consider a wavelength $\lambda = 5461 \times 10^{-8}$ cm and focal length $f = 10$ cm. Substituting in equation (107) we find

$$\left(\frac{r}{f}\right)_{max} = 0.67 \left(\frac{4\lambda}{2\pi f}\right)^{1/4} = 0.67 \left(\frac{4 \times 5461 \times 10^{-8}}{2\pi \times 10}\right)^{1/4}$$

$$\left(\frac{r}{f}\right)_{max} \simeq 0.03 \qquad (108)$$

Equation (108) specifies the aperture limit in the frequency plane F' to assure an error limit of less than 2% due to neglecting phase variations. We note that this phase limit restricts the frequency aperture to approximately one - fifth of the previous value of $\frac{r}{f}_{max} = 0.14$ which was sufficient for the linearization and amplitude approximations. The corresponding frequency limit is given as

$$p_{max} = \frac{1}{\lambda}\left(\frac{r}{f}\right)_{max} \simeq \frac{0.03}{5461 \times 10^{-8}} = 550 \text{ cycles/cm} \qquad (109)$$

Thus for spatial frequencies less than 550 cycles / cm, neglecting the phase term to obtain the Fourier transform representation of equation (91) introduces an error of no more than 2% in terms of the form of the third term in the image intensity of equation (101). Again we can point out that for most practical cases the frequency capability of present input techniques restricts the possible frequencies to a lower value than that specified by equation (109).

We have shown how equations (103) and (105) are used to determine the error E_ϕ for any frequency plane aperture with a radius defined by $\left(\frac{r_{max}}{f}\right)$. Further we have shown for a particular case ($\lambda = 5461$ Å and $f = 10$ cm) that the limit $\left(\frac{r}{f}\right)_{max} = 0.03$ provides an accuracy within 2% for the terms in which the phase variation appears. It has also been noted that this phase approximation requires a tighter restriction on the maximum frequency terms. In fact, for examples used the maximum frequency is one - fifth of that allowed for an accurate amplitude approximation. Of course, this further restriction of the frequency range of interest will also improve the accuracy of the amplitude approximation.

We can refer back to Figure 12 to consider the amplitude error E_A for the values $\frac{d}{f} = 1$ and $\frac{r_{max}}{f} = 0.03$ assuming $\frac{r_{1max}}{f} = \frac{1}{5}$. The point corresponding to $\frac{d}{f} = 1$ and $\frac{r_{max}}{f} = 0.03$ is located below the curve corresponding to $E_A = 0.005$. Therefore, the further restriction on $\frac{r_{max}}{f}$ required for the phase approximation reduces the error in the amplitude approximation to a value less than 0.5%. This result shows that the restriction we have considered in this section not only provides a Fourier transform relation which is accurate in phase, but also improves the accuracy of the amplitude approximations previously considered.

Within the limits presented in this section, the two - lens optical imaging process can be described by the two equations:

$$A(x, y) = \left[\frac{-ie^{i2kf}}{2\lambda f}\right] F(p, q) = \left[\frac{-ie^{i2kf}}{2\lambda f}\right]\iint S(x_1, y_1)\, e^{-i2\pi(px_1+qy_1)}\, dx_1\, dy_1 \qquad (110)$$

$$O(x_2, y_2) = e^{ikR_2}\left[\frac{-fe^{i2kf}}{2(f+d_2)}\right]\iint F(p, q)\, e^{-i2\pi(px_2+qy_2)}\, dp\, dq \qquad (111)$$

In practice only the variations in amplitude are of interest and the constant factors within the brackets are dropped

$$A(x, y) = F(p, q) = \iint S(x_1, y_1)\, e^{-i2\pi(px_1+qy_1)}\, dx_1\, dy_1 \qquad (112)$$

$$O(x_2, y_2) = e^{ikR_2} S(-x_2, -y_2) = e^{ikR_2}\iint F(p, q)\, e^{-i2\pi(px_2+qy_2)}\, dp\, dq \qquad (113)$$

Equations (112) and (113) represent the form of the optical Fourier transform representation commonly used. These equations describe the relative amplitude and phase variations of spectrum $A(x,y)$ and image $O(x_2, y_2)$. Note that the phase term e^{ikR_2} is retained in equation (113). This factor has no effect on the image intensity since multiplication by the complex conjugate eliminates this term. However, if the image $O(x_2, y_2)$ is to be processed further by another lens, the effect of the phase factor e^{ikR_2} must be considered. In such cases, our criteria for neglecting the variation in phase due to the factor e^{ikR_1} must also be re - evaluated since the criteria developed above was based on image intensity effects.

OPTICAL CORRELATOR SYSTEMS

We will now consider a three lens optical system as shown in Figure 16. In this system the signal plane F is assumed to be in the front focal plane of lens L_1 and each of the other lenses (L_2 and L_3) is located so that its front focal plane coincides with the back focal plane of the preceding lens. With this configuration the amplitude distribution corresponding to the input signal to each lens is the output signal in the back focal plane of the preceding lens and is a focal length in front of the lens. This location of the signal planes provides the advantage of eliminating the phase terms dependent upon the distance fron the lens to the input plane as discussed in relation to equations (88) and(89). The optical system in Figure 16 consists of a two lens imaging system as discussed in the preceding section followed by a third lens which produces a Fourier transform of the light amplitude distribution of the image. As pointed out at the end of the last section, the processing of the image $O(x_2, y_2)$ by an additional lens involves its amplitude rather than the intensity; therefore, the phase effects of each lens will be considered. Throughout this section we will assume that the aperture limitations are sufficiently restrictive so that the linear frequency and amplitude approximations developed earlier are valid. The focal lengths of the three lenses

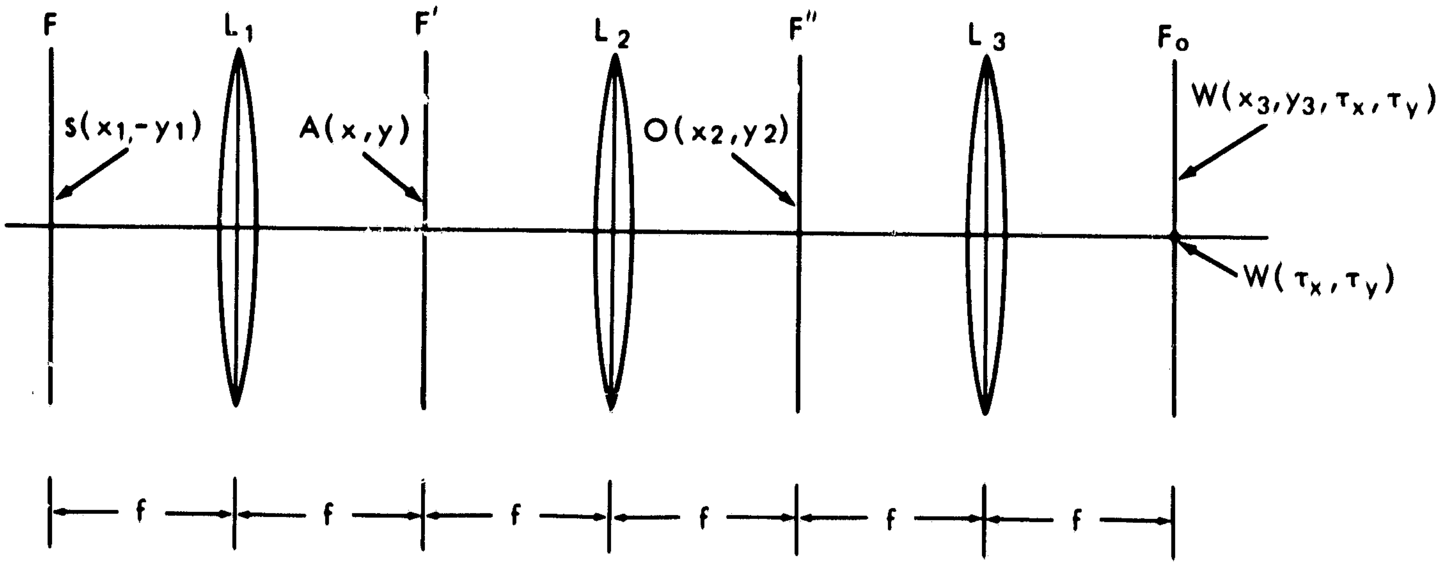


Figure 16—Optical correlator system.

are assumed equal to simplify our analysis; in the general case, unequal focal lengths would introduce magnification or demagnification.

In the optical system of Figure 16, lens L_1 produces a light amplitude distribution in its back focal plane F' which is proportional to the Fourier transform of the input signal $S(x_1, y_1)$ except for a multiplicative phase factor. As discussed in reference to equations (112) and (113) we will drop all constant factors and retain only terms which vary with respect to the coordinates in the four signal planes of interest (F, F', F'', F_0). Using only the variable exponential in equation (89) for K_1, the amplitude in the F' plane is given by equation (75) which can be written as

$$A(x, y) = e^{ikf(r/f)^4/4} F(p, q) = e^{ikf(r/f)^4/4} \iint S(x_1, y_1)\, e^{-i2\pi(px_1+qy_1)}\, dx_1\, dy_1 \qquad (114)$$

For our development of a correlator it is advantageous to introduce notation for the signal $S(x_1, y_1)$ which accounts for displacement of the signal from some reference position. Referring to Figure 17, we can consider the displacement of a signal point A to the new position A'. If A is a point of the signal $S(x_1, y_1)$, the light amplitude at A is $S(x_A, y_A)$. Since A' is the same signal point as A (it has only been moved), the light amplitude at A' must also be $S(x_A, y_A)$. The coordinates of the point A' are $x_1 = x_A + \tau_x$ and $y_1 = y_A + \tau_y$. Thus our notation for the signal must be such that if we substitute the coordinates x_1, y_1 for the point A' we obtain $S(x_A, y_A)$. The required notation is $S(x_1 - \tau_x, y_1 - \tau_y)$ as can be seen by substituting the values of x_1 and y_1 for each of the points A and A' . In either case the signal amplitude is $S(x_A, y_A)$. Using this new notation for a signal, equation (114) can be rewritten as

$$A(x, y) = e^{ikf(r/f)^4/4} F(p, q) = e^{ikf(r/f)^4/4} \iint S(x_1 - \tau_x, y_1 - \tau_y)\, e^{-i2\pi(px_1+qy_1)}\, dx_1\, dy_1 \qquad (115)$$

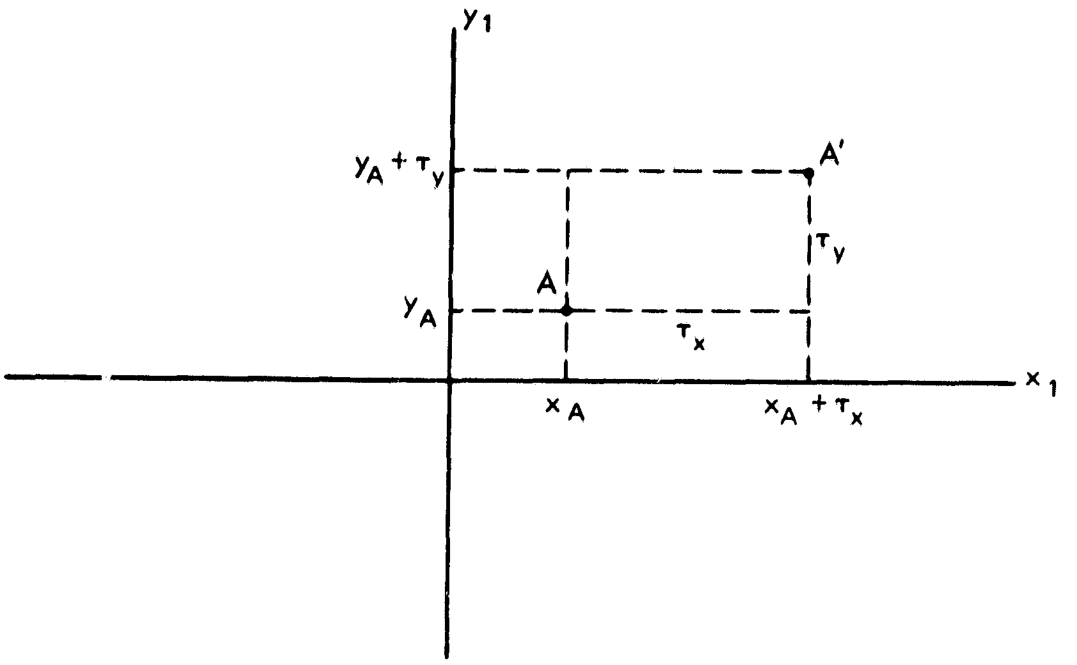


Figure 17—Displacement of a signal point in the input plane F.

The displacements τ_x and τ_y are positive when the displacement is in the positive x_1 and positive y_1 directions.

In equation (115) the function F(p, q) represents the Fourier transform of the displaced function $S(x_1 - \tau_x, y_1 - \tau_y)$. Thus the F(p, q) in equation (115) includes all the information regarding the signal including its displacement. From Fourier transform theory, the transform corresponding to a displaced signal such as in equation (115) differs from the transform F(p, q) of the undisplaced signal of equation (114) by an exponential phase term, $e^{-i2\pi(\tau_x p + \tau_y q)}$. This principle need not concern us any further; it is pointed out only to emphasize that the F(p, q) in equation (115) corresponds to the displaced signal $S(x_1 - \tau_x, y_1 - \tau_y)$.

The amplitude distribution given by equation (115) appears in the plane F′ and represents the input signal to the lens L_2. Lens L_2 performs a Fourier transform operation on A(x, y) and the image amplitude in the plane F″ is given by

$$O(x_2, y_2) = \phi(x_2, y_2)\iint e^{ikf(r/f)^4/4} F(p, q)\, e^{-i2\pi(px_2 + qy_2)}\, dp\, dq \qquad (116)$$

Equation (116) corresponds to equation (81) except that only the variable terms of K_1 and K_2 have been retained. The exponential appearing in the integrand corresponds to the variable term in K_1 as discussed above. The function $\phi(x_2, y_2)$ in front of the integral represents the variable part of K_2 . From the definition of K_2 given under equation (76) the variable part of K_2 is obtained from the term e^{ikR_2} where

$$R_2 = \frac{f^2 + fd_2 + r_2^2}{\left(f^2 + r_2^2\right)^{1/2}} \quad \text{and} \quad r_2^2 = x_2^2 + y_2^2 \tag{117}$$

Since R_2 has the same form as R_1, R_2 can be expanded in the form of equation (86)

$$R_2\left(x_2, y_2\right) = \left(f + d_2\right) + \left(f - d_2\right)\left[1 - \frac{1}{\left(1 + \frac{r_2^2}{f^2}\right)^{1/2}}\right] + 2f\left[\frac{1 + \frac{r_2^2}{2f^2}}{\left(1 + \frac{r_2^2}{f^2}\right)^{1/2}} - 1\right] \tag{118}$$

The first term and the - 1 term in the last bracket of equation (118) are constant and can be dropped since we are interested only in the variable part. The second term vanishes since $d_2 = f$ in the system we are considering. Thus the only variable term in equation (118) is the fraction in the brackets of the third term. The function $\phi(x_2, y_2)$ is therefore given by

$$\phi\left(x_2, y_2\right) = e^{i2kf\left[\left(1+r_2^2/2f^2\right)\left(1+r_2^2/f^2\right)^{-1/2}\right]} \tag{119}$$

The variable part of K_2 given by equation (119) was derived from the complete expansion of R_2 rather than from an approximate expansion analogous to equations (88) and (89), since the aperture restrictions necessary for the validity of (88) or (89) would require a signal and image aperture much smaller than that normally desired in optical systems. For an image aperture defined by $\left(\frac{r_2}{f}\right)_{max} = 0.14$, and $f = 10$ cm, and $\lambda = 5461 \times 10^{-8}$ cm , the phase term $\phi(x_2, y_2)$ can introduce phase shifts as great as 38π radians (19 cycles). It was pointed out that the phase approximation of equation (89) was accurate within 2%. For the image aperture considered here this phase inaccuracy can be of the order of 0.4 cycles. This magnitude of phase error may not be negligible and therefore the more complete exponential was used in defining $\phi(x_2, y_2)$ by equation (119).

Returning to the image amplitude distribution $O(x_2, y_2)$ given by equation (116), we will change the notation to take into account the possibility of image displacement corresponding to the signal displacement considered previously. In the last two sections it was pointed out that the imaged amplitude $O(x_2, y_2)$ corresponds to an inverted replica of the input signal $S(x_1, y_1)$. This inverted property of the image applies to the image motion as well. That is, if the signal is displaced in the positive x_1 and y_1 direction, the image is displaced in the negative x_2 and y_2 directions. Thus, if $O(x_2, y_2)$ corresponds to the inverted image of $S(x_1, y_1)$, the displaced image corresponding to

$S(x_1 - \tau_x, y_1 - \tau_y)$ is obtained simply by reversing the sign of the displacement to obtain $O(x_2 + \tau_x, y_2 + \tau_y)$. Using the displacement notation for the image amplitude distribution, equation (116) can be rewritten as

$$O\left(x_2 + \tau_x, y_2 + \tau_y\right) = \phi\left(x_2, y_2\right) \iint e^{ikf(r/f)^4/4} F(p, q)\, e^{-i2\pi(px_2 + qy_2)}\, dp\, dq \qquad (120)$$

The final lens L_3 in Figure 16 operates on the light amplitude distribution appearing in its input plane F''. For an optical correlator operation a reference signal $R(x_2, y_2)$ is inserted into the plane F'' in the form of an amplitude transmission function of a photographic transparency. In this case the light amplitude distribution operated on by lens L_3 is that which appears on the output side of the reference transparency. This light amplitude is given by the product of the incident light amplitude $O(x_2 + \tau_x, y_2 + \tau_y)$ and the reference transmission function $R(x_2, y_2)$. Thus the light amplitude distribution W in the output plane F_0 is given by the equation

$$W\left(x_3, y_3, \tau_x, \tau_y\right) = e^{ikf(r_3/f)^4/4} \iint R\left(x_2, y_2\right) O\left(x_2 + \tau_x, y_2 + \tau_y\right) e^{-i2\pi(sx_2 + ty_2)}\, dx_2\, dy_2 \qquad (121)$$

where

$$s = \frac{x_3}{\lambda f}, \qquad t = \frac{y_3}{\lambda f}, \qquad r_3^2 = x_3^2 + y_3^2$$

Since we are considering a system which terminates at the F_0 plane, the intensity will be detected, measured, or recorded in the F_0 plane. The intensity in the output plane is given by the product of equation (12. and its complex conjugate. The complex conjugate product of the exponential in front of the integral results in the cancellation of the exponential. Thus, we can drop the exponential in equation (121) since it will not affect the detected intensity output. Equation (121), therefore, can be simplified to

$$W\left(x_3, y_3, \tau_x, \tau_y\right) = \iint R\left(x_2, y_2\right) O\left(x_2 + \tau_x, y_2 + \tau_y\right) e^{-i2\pi(sx_2 + ty_2)}\, dx_2\, dy_2 \qquad (122)$$

Finally, if we consider only the point located at the intersection of the optical axis with the plane F_0 (back focal point of L_3), $s = t = 0$ (i.e. $x_3 = y_3 = 0$) and equation (122) reduces to

$$W(\tau_x, \tau_y) = \iint R\left(x_2, y_2\right) O\left(x_2 + \tau_x, y_2 + \tau_y\right) dx_2\, dy_2 \qquad (123)$$

where

$$W(\tau_x, \tau_y) = W(x_3 = 0, y_3 = 0, \tau_x, \tau_y)$$

Equation (123) corresponds to a two-dimensional correlation function which implies that the light amplitude at the back focal point $(x_3 = y_3 = 0)$ of the lens L_3 is given by the cross correlation of the reference $R(x_2, y_2)$ and the image amplitude $O(x_2, y_2)$. Thus as the input signal is displaced the variation of the light amplitude $W(\tau_x, \tau_y)$ corresponds to the variation of the correlation function with respect to the displacements τ_x and τ_y. Note that the correlation function defined by equation (123) involves the image amplitude $O(x_2, y_2)$ which is inverted with respect to the input signal $S(x_1, y_1)$. Therefore, if $R(x_2, y_2)$ is not a symmetrical function, it must be oriented correctly with respect to the image $O(x_2, y_2)$ rather than with respect to the input signal $S(x_1, y_1)$.

We shall briefly consider the implication of the steps from equation (122) to equation (123). This step in our derivation was accomplished by stating that we would consider only the single point in the output plane F_0 which lies on the optical axis (i.e. $x_3 = y_3 = 0$). In practice it is physically impossible to isolate a single point. The best attempt we can make is to restrict our light measurement or detection to a small area about the selected point. The light amplitude at points within this area (except for the one point on the optical axis) is given by equation (122) rather than (123). The light amplitude distribution will not be uniform over the finite area of measurement due to the phase variation involved in the integral of equation (122). For example, if we use a pinhole aperture 10 microns in diameter to define our detection area, the phase term in equation (122) can vary as much as 4π radians (2 cycles) over the range of the image aperture (assuming $r_{2max} \sim 0.14f$, $f = 10$ cm, $\lambda = 5461 \times 10^{-8}$ cm). The effects of the phase term in equation (122) is to reduce the light amplitude at points off axis since the contributions to the integral are not in phase. Therefore, the actual light available through a pinhole aperture located in the F_0 plane at $x_3 = y_3 = 0$ will be less than that found by assuming the light amplitude given by equation (123) appears at all points within the pinhole aperture. We will not consider this problem any further here since the analysis would depend on the type of photo-detector or measurement technique used. We will assume that the variations involved are small enough so that any measurement will yield values proportional to the square of the amplitude given by equation (123).

As pointed out above the correlation function defined by equation (123) involves the image amplitude $O(x_2, y_2)$ rather than the single amplitude $S(x_1, y_1)$. As defined by equation (120) the image amplitude contains phase terms not present in the signal. A correlation operation can be performed based on the image as given by equation (123); however, the reference signal $R(x_2, y_2)$ would have to be selected in terms of the image $O(x_2, y_2)$ including the phase terms. The correlation function obtained would correspond to a distorted signal rather than the actual signal $S(x_1, y_1)$. The presence of distortion due to the phase terms in equation (120), therefore, complicates the analysis and determination of the correlation process. For example, the image amplitude $O(x_2, y_2)$ will be complex (phase variation as well as amplitude) and for complete correlation a complex reference signal is required. Such reference transparencies are difficult to produce. The phase distortions are commonly neglected and a reference signal is selected based on an ideal image (no distortion) of the input signal. We will now proceed to analyze such a system to determine the effects of the undesirable phase terms present in our equations.

Let us consider a signal which would produce an ideal image amplitude defined by

$$O(x_2 + \tau_x) = \sum_n B_n \cos 2\pi p_n (x_2 + \tau_x) \tag{124}$$

Equation (124) defines a signal image composed of a series of cosine harmonics in one dimension. A one - dimensional signal has been chosen to simplify our analysis. Referring to equation (120) we find that each frequency term in the image has a phase term $e^{ikf(\lambda p_n)^4/4}$ associated with it and the image also has a phase term $\phi(x_2, y_2)$ associated with it. Thus for the actual image we would have

$$O(x_2 + \tau_x) = \phi(x_2) \sum_n B_n e^{ikf(\lambda p_n)^4/4} \cos 2\pi p_n (x_2 + \tau_x) \tag{125}$$

We can consider a reference signal without phase given by

$$R(x_2) = \sum_m R_m \cos 2\pi p_m x_2 \tag{126}$$

The reference signal $R(x_2)$ defined by equation (126) has been selected to have the same cosine harmonics $(m = n)$ as the imaged signal being considered. Note that the reference signal defined by equation (126) does not contain the phase terms present in equation (125). The product of reference and image for the ideal image of equation (124) is given by

$$R(x_2)\, O(x_2 + \tau_x) = \sum_{n,m} B_n R_m \cos 2\pi p_m x_2 \cos 2\pi p_n (x_2 + \tau_x) \tag{127}$$

The product of reference and image for the actual image of equation (125) is given by

$$R(x_2)\, O(x_2 + \tau_x) = \phi(x_2) \sum_{n,m} B_n R_m e^{ikf(\lambda p_n)^4/4} \cos 2\pi p_m x_2 \cos 2\pi p_n (x_2 + \tau_x) \tag{128}$$

Substituting equation (127) and (128) into equation (123), we obtain for the correlation function of the ideal image:

$$W(\tau_x) = \sum_{n,m} B_n R_m \int \cos 2\pi p_m x_2 \cos 2\pi p_n (x_2 + \tau_x)\, dx_2 \tag{129}$$

and for the correlation functions of the actual image:

$$W(\tau_x) \qquad \sum_{n,m} B_n R_m e^{ikf(\lambda p_n)^4/4} \int \phi(x_2) \cos 2\pi p_m x_2 \cos 2\pi p_n (x_2 + \tau_x) dx_2 \qquad (130)$$

Comparing equations (129) and (130) we find that the term $e^{ikf(\lambda p_n)^4/4}$ affects the phase of each term in the double summation. From our previous discussion of frequency limitations with respect to this phase term we can show that applying the limitation developed for imaged intensity limits the phase variation of this term to approximately 7°. In summing terms which are not in phase the result will be less than summing the same terms in amplitude only. Thus the presence of the phase term $e^{ikf(\lambda p_n)^4/4}$ has the effect of reducing the value of $W(\tau_x)$ in equation (130) as compared to equation (129). However, since the maximum phase will be about 7° the difference due to this term will be small. The phase term $\phi(x_2)$ appears in the integral of each term in the sum and has the same effect on the integral (can be considered as summation) as the phase term discussed above had on the summation. However, as discussed above the phase variations of $\phi(x_2)$ ranges over 19 cycles and the effect on the value of the integral will be correspondingly greater. The actual magnitude of the reduction in $W(\tau_x)$ due to these phase terms is difficult to evaluate in general since the reduction will depend on the form of the signals involved. However, from our discussion here it is apparent that the actual correlation function observed will be smaller in amplitude than that predicted using an ideal image. This result is obvious if we consider that the presence of the phase terms in the actual image produce a mismatch between the signal and reference and therefore the correlation will be reduced. The effects of these phase terms can be reduced by further restricting the frequency range p_{max} (or r_{max}) and signal and image aperture size which would limit the variation of the phase terms. We will not proceed with an analysis of the required limitations since the analysis will depend to a large extent on the type of signals involved and the correlation results desired. Here we have developed the equations necessary for such an evaluation and hopefully have pointed out the significance of the various effects which appear in an optical system.

PHASE CORRECTIONS

In the last few sections we have discussed the effects of undesirable phase terms in optical systems and have demonstrated that these effects can be minimized by restricting the size of signal apertures and the spectral range of the signals. An alternative approach can be pursued by inserting phase corrections into the optical system. Such phase corrections can be implemented by inserting sheets or plates of transparent materials whose thickness or index of refraction has variations which introduce phase terms opposite in sign to those introduced by the system.

The basic equation representing the Fourier transform operation of a lens was given by equation (45) as

$$A(x, y) = \frac{-i e^{ikR(x,y)}}{\lambda(f+d)} \iint A'(x_1, y_1) e^{-i2\pi(px_1 + qy_1)} dx_1 \, dy_1 \qquad (45)$$

Rewriting this equation retaining only the variable part of the terms outside the integral we obtain

$$A(x, y) = \phi(x, y) \iint A'\left(x_1, y_1\right) e^{-i2\pi(px_1+qy_1)}\, dx_1\, dy_1 \tag{131}$$

where

$$\phi(x, y) = e^{i2kf\left[(1+r^2/2f^2)(1+r^2/f^2)^{-1/2}\right]} \quad \text{assuming} \quad d = f$$

as derived in equation (119). As pointed out in all our discussions the phase term $\phi(x, y)$ destroys the simple Fourier transform representation of lens focussing properties since the integral part of equation (131) corresponds to a Fourier transform by itself. Let us consider inserting a phase correction plate into the back focal plane of a lens with transmission properties given by

$$P(x, y) = A_0\, e^{iC} \phi^*(x, y) \tag{132}$$

where

$$A_0 = \text{constant}$$

$$C = \text{constant}$$

$$\phi^*(x, y) = e^{-i2kf\left[(1+r^2/2f^2)(1+r^2/f^2)^{-1/2}\right]}$$

The light amplitude distribution appearing at the output side of the plate will be

$$A(x, y)P(x, y) = \iint A'\left(x_1, y_1\right) e^{-i2\pi(px_1+qy_1)}\, dx_1\, dy_1 \tag{133}$$

where the constant term $A_0\, e^{iC}$ has been dropped and $\phi(x, y)\, \phi^*(x, y) = 1$. Thus by inserting a phase plate with transmission properties given by equation (132) in the back focal plane of each lens in an optical system the phase terms are eliminated. From the definition given by equation (132) we find that the phase correction depends on the focal length f of the lens and the wavelength $\lambda \left(k = \frac{2\pi}{\lambda}\right)$ of the light. The phase correction is not dependent on the signal used and therefore a phase plate can be made for the lens and wavelength to be used in the system. Of course, the correction of phase by this method requires an accurate technique for producing the phase plate and positioning the plate in the optical system. In any case, we have shown that the elimination of undesirable phase terms is possible at least in theory. Any inaccuracies in production or location of the phase plate may be acceptable as long as the phase terms are appreciably less than before the plate was introduced. Assuming the phase plate is an accurate representation of the transmission function of equation

(132), the Fourier transform relation of equation (133) will be valid. With the relationship given by equation (133) the operation of spectrum analyzer, imaging and optical correlation systems can be described by the ideal cases used in the respective discussions and no undesirable phase terms appear in the equations.

PHASE TERMS WHEN $d \neq f$

In our consideration of phase terms we considered the special case of an input plane coincident with the front focal plane of a lens $(d = f)$. This special case was chosen to eliminate the phase effects of a term proportional to $(f - d)$. From equation (118) we can write a complete expression for the variable part of the exponential term e^{ikR} as

$$\phi(x, y) = e^{ik(d-f)(1+r^2/f^2)^{-1/2}} e^{i2kf(1+r^2/2f^2)(1+r^2/f^2)^{-1/2}} \qquad (134)$$

Equation (134) reduces to the form of equation (119) when $d = f$. If we can restrict our consideration to a rather limited range in the back focal plane of a lens, we have shown that equation (134) can be given to a good approximation in the form of equation (88)

$$\phi(x, y) = e^{ik(f-d)(r/f)^2/2} e^{ikf(r/f)^4/4} \qquad (135)$$

We can consider equation (135) as a representation of the phase in a back focal plane containing a frequency spectrum while equation (134) is a more accurate representation which applies in a back focal plane containing an image of the input signal. This application of equation (134) and (135) is based on the relatively larger apertures commonly used in the signal and image planes.

In the systems which we have considered here the complete phase variations as given by equation (134) appear only in the correlator system. This can be seen by noting the presence of $\phi(x_2, y_2)$ in equation (120). Since this phase factor is expressed in terms of the coordinates x_2 and y_2 of the image phase we cannot use a very restrictive aperture limitation without severely effecting our signal handling capability. Therefore, the approximation of equation (135) will not be valid and $\phi(x_2, y_2)$ in equation (120) will have the form of equation (134) with d_2 and r_2 replacing d and r respectively

$$\phi(x_2, y_2) = e^{ik(d_2-f)(1+r_2^2/f^2)^{-1/2}} e^{i2kf(1+r_2^2/2f^2)(1+r_2^2/f^2)^{-1/2}} \qquad (136)$$

where $r_2^2 = x_2^2 + y_2^2$ and d_2 is the distance from the spectrum plane F' to the lens L_2 (see Figure

16). In the sample correlation function of equation (130) we can see that the additional phase term dependent on $(d_2 - f)$ will increase the effect of $\phi(x_2)$ on the integrals. In practice, a system would be specified on the basis of locating lens L_2 so that $d_2 = f$. However, the exact positioning of the lenses in an optical system is obviously a practical impossibility. Thus the additional phase term containing $(d_2 - f)$ represents the phase distortion introduced by inaccuracies in the implementation of the system. Since the quantity $(d_2 - f)$ represents an inaccuracy its value will usually be undetermined. Therefore, the first term in equation (136) represents an undetermined phase error in the optical correlator system. If a guess or estimate of the tolerances in the system can be made, this error term can be used to determine the maximum distortion of the correlation function by analysis similar to that implied by equation (130).

Since the variation of the phase term containing $(d_2 - f)$ in equation (136) is not known specifically, the elimination of this term by a phase correction plate is not possible. Thus in a system containing phase correction plates only the second term of equation (136) can be eliminated. In such systems $\phi(x_2, y_2)$ is completely given by the position error term

$$\phi(x_2, y_2) = e^{ik(d_2-f)(1+r_2^2/f^2)^{-1/2}} \qquad (137)$$

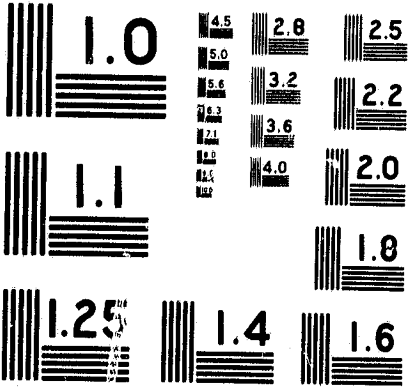
The distortion of the correlation function in a phase corrected system is therefore completely dependent upon the positioning errors. Again referring to the sample of equation (130), $\phi(x_2)$ would be given in the form of equation (137). The phase term in front of the integral of equation (130) would also be replaced by an error term from an expression such as equation (135) as will be discussed below.

The phase term given by equation (135) represents the variable part of equation (88). The exponential dependent on $(f - d_1)$ represents an error term due to lens positioning. To account for this error the complete phase approximation of equation (135) must be used in place of the K_1 exponential of equation (89). Thus the error phase term will appear throughout our previous analysis wherever we have used the K_1 term.

We considered the effect of the K_1 phase term on the image intensity and on the correlation function in earlier sections. In our correlator discussion the variable phase term of K_1 appears in the integral used to define the image amplitude distribution in equation (120). To account for errors in placement of lens L_1 (see Figure 16) the exponential $e^{ikf(r/f)^4/4}$ in equation (120) must be replaced by a phase term of the form of equation (135) which can be written

$$\phi(x, y) = e^{ik(f-d_1)(r/f)^2/2} e^{ikf(r/f)^4/4} \qquad (138)$$

Referring to our sample correlation of equation (130) the phase term given by equation (138) will replace the $e^{ikf(r/f)^4/4}$ term in front of the integral. The error phase term has the affect of adding


1.0
4.5
5.0
5.6
6.3
7.1
8.0
9.0
10.0
2.8
3.2
3.6
4.0
2.5
2.2
2.0
1.1
1.8
1.25
1.4
1.6

an additional variation to the phase of the terms in the summation. For a phase corrected system the $e^{ik(r/f)^4/4}$ term is eliminated and the undesirable phase difference of terms in the summation will be dependent only on the accuracy of the system implementation.

In our discussion of imaging systems we defined a factor m by equation (103) and developed a method for determining the accuracy of the image intensity based on this parameter. To extend this method to include the case for $d_1 \neq f$ we merely redefine m by the equation

$$m^4 = \frac{k}{2}\left(\frac{r}{f}\right)^2 \left[(f-d_1) + \frac{f}{2}\left(\frac{r}{f}\right)^2\right] \tag{139}$$

which is obtained from the exponents of the terms in equation (138). For d_1 less than f the limits on m defined in our previous discussion will apply to equation (139) for the maximum value of r. It is noted that since $(f - d_1)$ is a positive quantity when d_1 is less than f the required limit on $\frac{r_{max}}{f}$ will be less than that determined for the case $d_1 = f$. When d_1 is greater than f, $(f - d_1)$ is a negative quantity which would imply that the value of $\frac{r_{max}}{f}$ can be greater than that for the case $d_1 = f$. This is true except for cases in which d_1 is sufficiently greater than f so that for some value of $\frac{r}{f}$ less than $\frac{r_{max}}{f}$ the value of equation (139) is greater in absolute value than for $\frac{r_{max}}{f}$. That is, since $(f - d_1)$ is negative the right side of equation (139) is zero at $r = 0$, becomes negative as r increases until it reaches a maximum negative value and then increases to positive values. Depending on the value of d_1 and the limit $\frac{r_{max}}{f}$ it is possible that the phase at the maximum negative value is greater than that at the aperture limit $\frac{r_{max}}{f}$. In such cases, the maximum negative value must be considered rather than the end value at $\frac{r_{max}}{f}$ and the aperture may have to be restricted to values below this maximum. In any case the brackets on the right side of equation (139) must be considered as an absolute value symbol when the quantity within is negative so that m will have real values. In other words, we are concerned with the magnitude of the phase variations and not the sign.

For systems with phase correction only the first term of equation (138) will remain and (139) will be simplified to

$$m^4 = \frac{k}{2}\left(\frac{r}{f}\right)^2 (f-d_1) \tag{140}$$

Equation (140) can then be used with the image intensity criteria developed earlier to consider the effects of positioning error for lens L_1 in phase corrected systems.

In most of the literature the phase corrected form of (138)

$$\phi(x, y) = e^{ik(f-d_1)(r/f)^2/2} \tag{141}$$

is used even though phase correction techniques may not be employed. This application of (141)

#2

requires that the frequency limitation be sufficient so that the $e^{ikf(r/f)^4/4}$ term can be neglected. This application also implies that the term $(f - d_1)$ is much greater than the maximum value of $\frac{f}{2}\left(\frac{r}{f}\right)^2$. If this condition does not hold the neglected term will contribute a phase comparable to that of equation (141) which would then be in error. Conversely, if $(f - d_1)$ is not greater than $\frac{f}{2}\left(\frac{r}{f}\right)^2$ and the $e^{ikf(r/f)^4/4}$ term is considered negligible, than the term given by equation (141) is also negligible since it is comparable to the neglected term.

In this section we have outlined the procedure for taking into account the additional phase term arising from inaccuracies in the positioning of lenses. It was pointed out that since these terms are due to inaccuracies they are generally not specified completely. The worse case, however, can be specified by estimating the maximum error in the position of a lens. From this extreme estimate the necessary aperture limitation or the evaluation of errors in the desired optical outputs can be determined for a worse case analysis. Unfortunately, due to the undetermined nature of these terms, phase correction cannot be used to eliminate their effects.

SUMMARY

The derivation presented in this report demonstrates that the Fourier transform representation of a focussed diffraction pattern is a reasonable approximation for describing the operation of coherent optical systems with lenses. The basic assumptions consisted of the ideal focal properties of a lens and the use of perfectly coherent light. Except for undesirable phase effects, it was demonstrated that the Fourier transform representation is obtained as a good approximation by imposing limitations on the size and frequency content of signals allowed. The phase terms can also be eliminated by aperture limitations; however, the restrictions are more severe. Depending on the application, a trade off must be made between the limitations required for elimination of undesired terms and the desired signal size and frequency content.

Techniques for evaluating the effects of the various approximations and for analyzing the operation of ideal optical systems have been presented. For specified signals and applications these expressions can be used to determine the theoretical errors in assuming ideal operations as is commonly practiced. The analysis presented is by no means complete; however, it is hoped that it is sufficiently detailed to provide a clear insight into the required approximations.

This report represents an initial step in the development of a detailed analysis of the capabilities of optical processing systems. Further studies are required to formulate complete criteria and analysis techniques for practical optical systems. Some of the important areas which must be considered include:

1. Lens aberrations
2. Coherence
3. Transmission properties of modulation media
4. Band - limited signal approximations

These areas were not treated in the analysis presented here since the initial study was restricted to ideal systems. The complexity of the mathematical formulation of optical patterns can be simplified somewhat by using the notation of communication theory[5]. Such methods are becoming quite useful in modern optics studies. The development of these techniques provides a means for avoiding the complicated mathematical formulations inherent in diffraction problems. However, any new formulations such as these must be considered in terms of the more rigorous formulation since the various approximations are basically the same in both formulations.

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APPENDIX

DERIVATION OF DIFFRACTION FORMULA

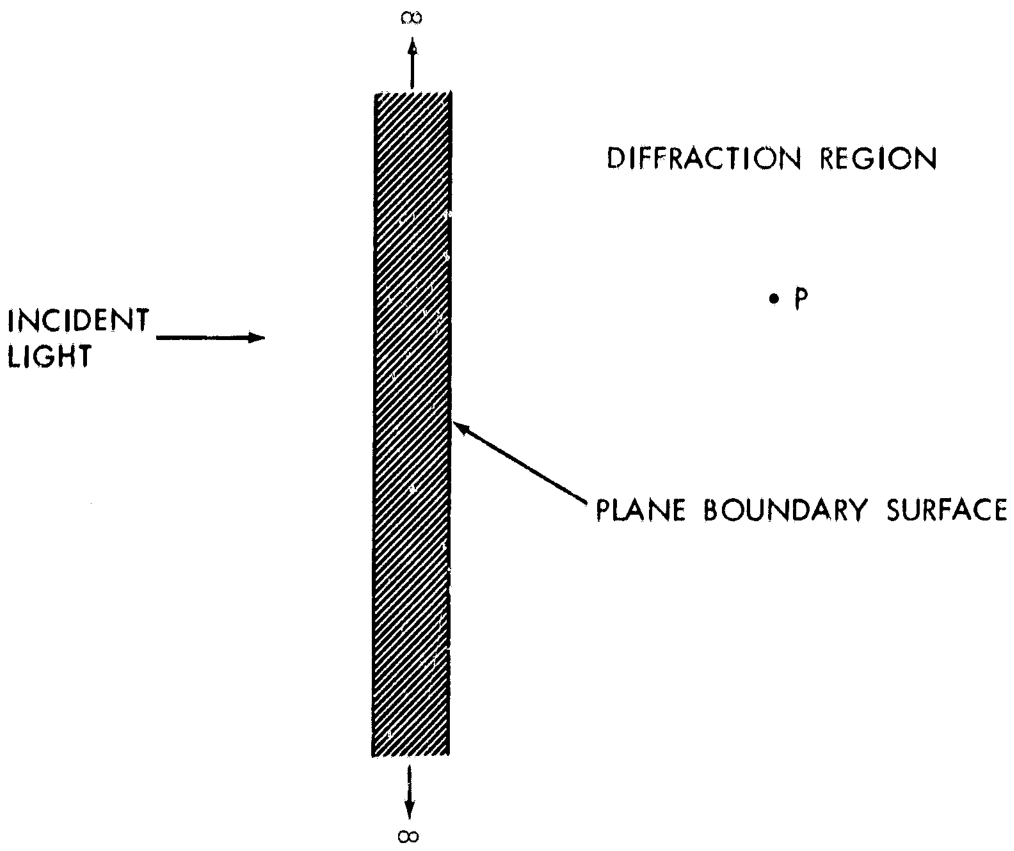

Figure A-1—Diagram of diffraction configuration.

We will restrict our discussion to the somewhat special case of diffraction at an infinite plane surface as diagrammed in Figure A - 1. The shaded area in the figure represents a cross - section of an infinite slab. The basic problem is to determine the electric field at any point P in the diffraction region which includes all points to the right of the plane boundary surface indicated in Figure A - 1. When the plane slab is not present, the electric field at any point P could be found simply by substituting the coordinates of P into the mathematical expression describing the light propagating from whatever light source may be present. In itself, finding a mathematical representation for a given light source is not a simple problem. The light radiated by a source is dependent upon the mechanism generating the light as well as the geometry of the source. In many cases it is assumed that a good approximation is obtained by considering ideal light sources which radiate spherical waves (point sources) or plane waves (point sources at an infinite distance).

Inserting a plane surface into the path of the light waves as shown in Figure A - 1 complicates our problem. Since the presence of the plane effects the propagation characteristics in space, the electric field at any point will now depend on the characteristics of both the light source and the plane. The characteristics of the plane depend on the type of material of which it is made and these characteristics usually vary from point to point in the surface. Thus we are confronted with the problem of determining the electric field in the presence of a surface which can have widely varying electrical properties from point to point. As the reader may already know, problems of this nature are very difficult and, in fact, very few diffraction problems have been solved rigorously. Fortunately, in many cases of practical interest results within experimental accuracy can be obtained by less rigorous techniques.

In order to implement a discussion of diffraction problems, we will now proceed to the derivation of a formula for diffraction at a plane surface. This result was first derived by Sommerfeld[6] in 1896 and as will be demonstrated is effectively a mathematical representation of Huygen's principle for the special case of a plane diffraction surface. The basic assumption we will start with is that the components of the electric field are known at every point on the right hand boundary surface of the plane slab (refer to Figure A-1). The methods for determining these field values are of no importance at this point; however, in many cases of interest assuming a multiplying factor representing the transmission properties of a thin material provides results in close agreement with experiment. For our present purpose, we will simply assume that the value of the electric field at every point on the plane boundary surface is known (i.e. can be found easily).

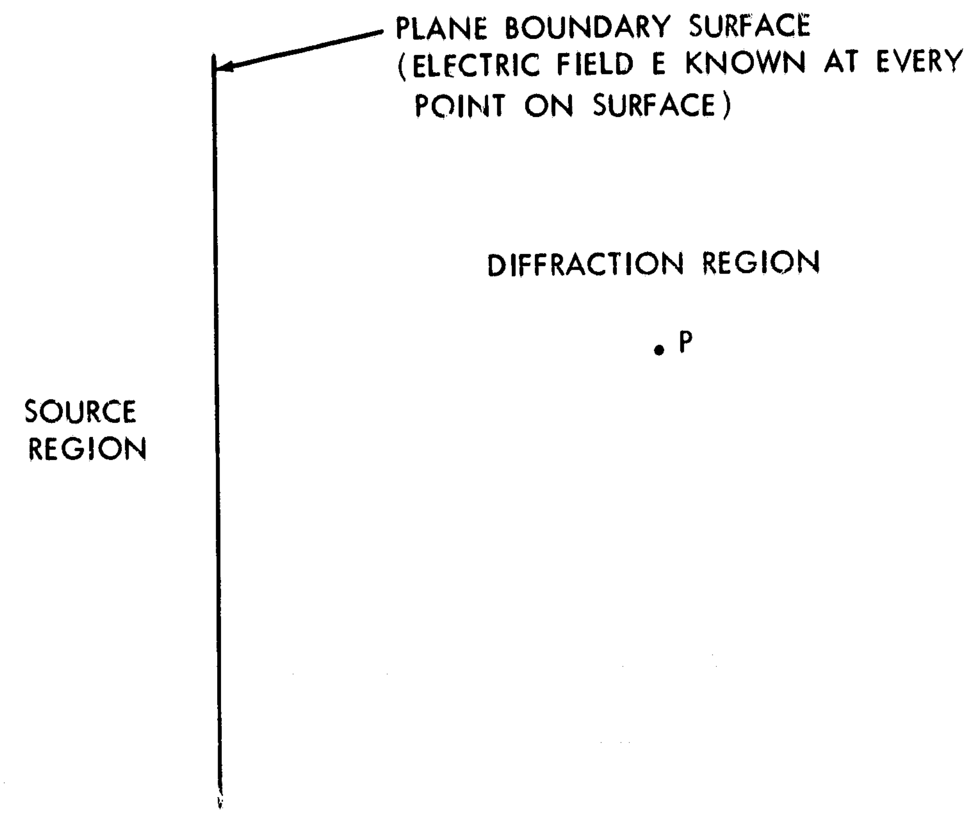

Figure A-2—Outline of diffraction problem.

Referring to Figure A-2, the problem we must solve can be stated as follows:

> Given the electric field at every point on an infinite plane boundary, what is the electric field at any point P in the diffraction region?

As shown in Figure A - 2, we define our diffraction region to be all space on the right side of the boundary surface (note that this region does not contain any light sources). In Figure A - 2, it is assumed that all light sources are to the left of the boundary surface and that the diffraction region includes all points to the right of the boundary. We can assume that our diffraction region is in free space (velocity of light is c in all directions) and specify that there are no electric currents or charges present in this region. Since the electric field is a vector quantity, its direction at any point is as important as its magnitude. As in the case of any vector we can consider the electric field in terms of its components in the x, y and z directions. To simplify our discussion we will assume that light waves are monochromatic, or vary in time at a single frequency. When necessary this discussion can be extended to the general case of non - monochromatic waves by considering each separate frequency component as described here and summing up all components.

Each component E (in x, y, z components) of the electric field of a monochromatic wave will satisfy the Helmhotz equation (time - independent wave equation) at every point P in free space which contains no electrical sources:

$$(\nabla^2 + k^2)E = 0 \qquad (1)$$

where

E = x, y, or z component of the electric field

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$k = \frac{\omega}{c} = \frac{2\pi}{\lambda}$$

ω = angular frequency

c = speed of light

λ = wavelength of light

Using the values for E on the plane boundary surface and the fact that E must satisfy equation (1) at every point P in our diffraction region, we can derive a formula for the electric field at P in terms of the values given on the surface.

We will introduce an arbitrary function V which also satisfies Helmholtz equation:

$$(\nabla^2 + k^2)V = 0 \qquad (2)$$

There are many functions which will satisfy equation (2); however, we will presently continue to use the symbol V and reserve the selection of a specific function until we determine a few additional characteristics of V which will allow us to accomplish our derivation. In terms of E and V, we can define two vectors $\vec{F}_1$ and $\vec{F}_2$ as

$$\vec{F}_1 = E \nabla V \tag{3}$$

$$\vec{F}_2 = V \nabla E \tag{4}$$

where

$$\nabla = \vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}$$

and ∇V and ∇E denote the gradient of V and E respectively. We now introduce Gauss's theorem:

$$\iiint\limits_{\text{volume}} \nabla \cdot \vec{F}\,dv = \iint\limits_{\text{surface}} \vec{F} \cdot d\vec{s} \tag{5}$$

where the volume integral on the left can be taken over any volume which does not contain discontinuities of the divergence of $\vec{F}$ $(\nabla \cdot \vec{F})$ and the surface integral on the right is over the surface which encloses the volume ($\vec{F}$ must be continuous on the surface so that the surface integral can be found). From (3) and (4) we obtain

$$\nabla \cdot \vec{F}_1 = \nabla \cdot E\nabla V = \nabla E \cdot \nabla V + E\nabla^2 V \tag{6}$$

$$\nabla \cdot \vec{F}_2 = \nabla \cdot V\nabla E = \nabla V \cdot \nabla E + V\nabla^2 E \tag{7}$$

From above we note that (6) and (7) must not have discontinuities within the volume of integration; therefore, E and V must have continuous first and second derivatives. Since we are considering a diffraction region free of electrical current and charge, E will meet this requirement for any volume in the diffraction region. Since we have not yet selected a specific V, we will note this requirement and be sure to satisfy it when selecting our V. Thus we can write equation (5) substituting (3), (4), (6) and (7):

$$\iiint \nabla \cdot \vec{F}_1\,dv = \iiint \{\nabla E \cdot \nabla V + E\nabla^2 V\}\,dv = \iint E\nabla V \cdot d\vec{s} \tag{8}$$

$$\iiint \nabla \cdot \vec{F}_2\,dv = \iiint \{\nabla V \cdot \nabla E + V\nabla^2 E\}\,dv = \iint V\nabla E \cdot d\vec{s} \tag{9}$$

We can substract (9) from (8)and noting that $\nabla E \cdot \nabla V - \nabla V \cdot \nabla E = 0$ we obtain:

$$\iiint \{E\nabla^2 V - V\nabla^2 E\}\,dv = \iint \{E\nabla V - V\nabla E\} \cdot d\vec{s} \tag{10}$$

This is Green's theorem and holds for any function E and V which have continuous first and second derivations in the region of integration. From equations (1) and (2) we know that $\nabla^2 E = -k^2 E$ and $\nabla^2 V = -k^2 V$ and therefore the bracket on the left hand side of equation (10) gives

$$E\nabla^2 V - V\nabla^2 E = -k^2 EV - \left(-k^2 EV\right) = 0 \qquad (11)$$

Since the integrand is zero as given in (11) the volume integral on the left hand side of (10) is zero and we can rewrite equation (10) as

$$\iint (E\nabla V - V\nabla E) \cdot d\vec{s} = 0 \qquad (12)$$

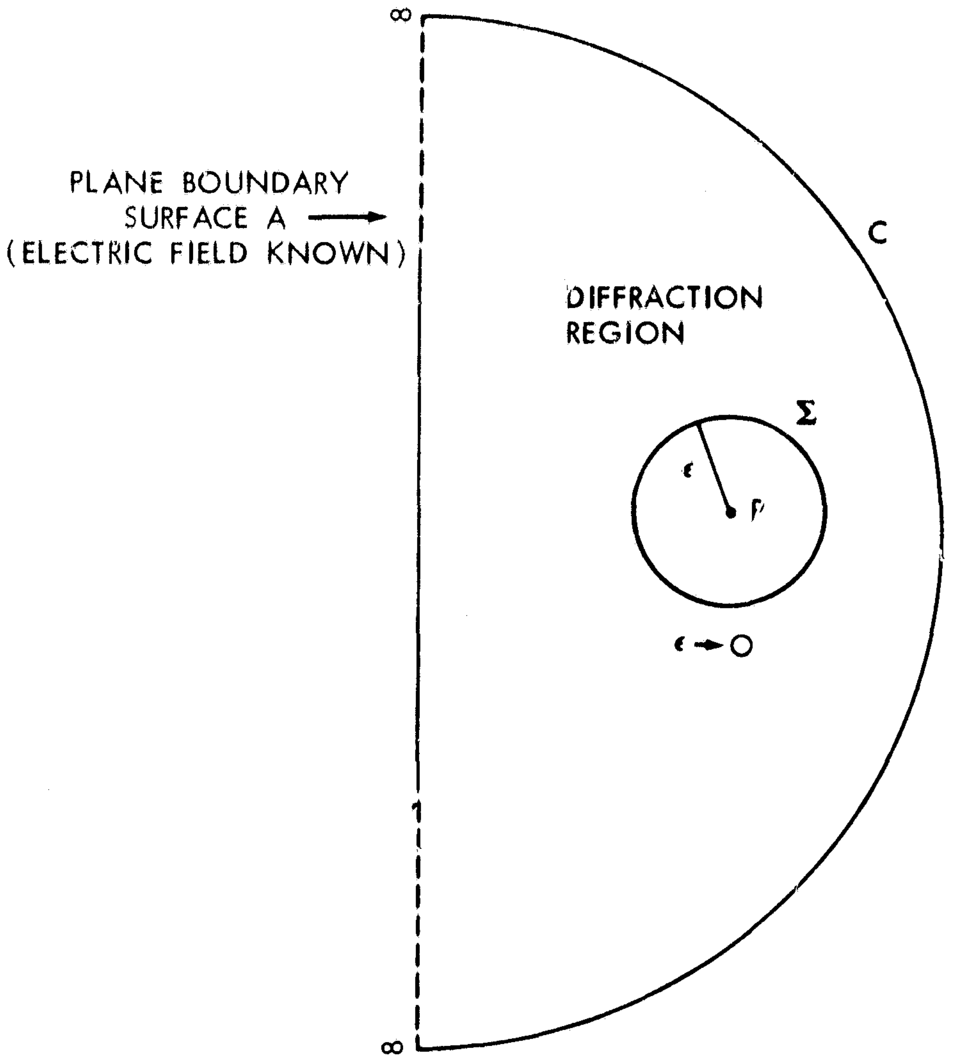

Figure A-3—Boundary surface enclosing all points except P.

The surface integral in equation (12) is to be taken over any closed surface which does not enclose discontinuous points. For our purposes we will choose the surface as indicated in Figure A-3. The outer surface consists of the infinite boundary plane (A) on which the electric field is known and a hemisphere (C) of infinite radius which connects the ends of the plane at infinity. We define an inner surface (Σ) as a sphere centered at P with radius ε. Taking A, C, and Σ as our closed surface, we have defined the volume between the sphere Σ and the outer surface A-C. If we take the limit as ε goes to zero, the point P will be the only point in the diffraction region outside the surface of integration. Thus the sphere Σ isolates the point P where we want to find the electric field. The surface integral of equation (12) can be written as the sum of the integrals over A, C, and Σ:

$$\iint_C \{E\nabla V - V\nabla E\} \cdot d\vec{s} + \iint_A \{E\nabla V - V\nabla E\} \cdot d\vec{s} + \iint_\Sigma \{E\nabla V - V\nabla E\} \cdot d\vec{s} = 0 \qquad (13)$$

The surface integral over the hemisphere at infinity can be eliminated through a physical argument given by Born and Wolf[4]. In practice a light wave starts at some time and since it propagates at a finite velocity (c in free space) must have an end. We can imagine the infinite hemisphere continually expanding in front of the light waves. In this way, the contribution of the wave on the hemisphere is zero since the light waves never reach the hemisphere. The integral over the surface C will therefore be zero and equation (13) can be written

$$\iint_A \{E\nabla V - V\nabla E\} \cdot d\vec{s} + \iint_\Sigma \{E\nabla V - V\nabla E\} \cdot d\vec{s} = 0 \qquad (14)$$

We will now take advantage of our freedom to select a function V in order to simplify equation (14). In the integral over the plane surface A we note that the term $V\nabla E$ requires the values of ∇E on the boundary surface. Since we do not know ∇E on A, we will require V to be zero on the boundary surface to eliminate this term. Equation (14) can then be written

$$\iint_A E\nabla V \cdot d\vec{s} + \iint_\Sigma \{E\nabla V - V\nabla E\} \cdot d\vec{s} = 0 \qquad (15)$$

To consider the integral over the sphere Σ, we will express the surface element $d\vec{s}$ in polar coordinates:

$$d\vec{s} = -\epsilon^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$$

where $\hat{r}$ is a unit vector in a radial direction away from the center at P and the minus sign is required by the convention that a surface normal is directed away from the enclosed volume. The integral over Σ can be written as

$$-\int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\phi \, \epsilon^2 \, (E\nabla V - V\nabla E) \cdot \hat{r} \qquad (16)$$

By vector identity $(E\nabla V - V\nabla E) \cdot \hat{r} = E\frac{\partial V}{\partial r} - V\frac{\partial E}{\partial r}$ and (16) can be written

$$-\int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\phi \left[E\left(\epsilon^2 \frac{\partial V}{\partial r}\right) - \epsilon^2 V \frac{\partial E}{\partial r}\right] \qquad (17)$$

Recall that we want to take the limit as ϵ goes to zero. The term $\epsilon^2 V \frac{\partial E}{\partial r}$ in (17) can be eliminated if we require V to satisfy the condition

$$\lim_{\epsilon \to 0} \epsilon^2 V = 0 \qquad (18)$$

at any point P in the diffraction region.

Since E has continuous first derivations $\frac{\partial E}{\partial r}$ will be finite and the conditioning given by (18) will give

$$\lim_{\epsilon \to 0} \epsilon^2 V \frac{\partial E}{\partial r} = 0 \qquad (19)$$

Expression (17) is then given as

$$- \int_0^{\pi} \sin \theta \, d\theta \int_0^{2\pi} d\phi \left\{ \lim_{\epsilon \to 0} E \left(\epsilon^2 \frac{\partial V}{\partial r} \right) \right\} \qquad (20)$$

Since E is the value of the field on the surface Σ and the surface Σ reduces to the point P when ϵ goes to zero, the limit in (20) can be written

$$\lim_{\epsilon \to 0} E \epsilon^2 \frac{\partial V}{\partial r} = E(P) \lim_{\epsilon \to 0} \epsilon^2 \frac{\partial V}{\partial r} \qquad (21)$$

where E(P) is the electric field at P. We will require that V satisfy the condition

$$\lim_{\epsilon \to 0} \epsilon^2 \frac{\partial V}{\partial r} = 1 \qquad (22)$$

at any point P in the diffraction region. The limit in the brackets of equation (20) is then simply E(P) and (20) gives

$$- \int_0^{\pi} \sin \theta \, d\theta \int_0^{2\pi} d\phi \, E(P) = - 4\pi E(P) \qquad (23)$$

Substituting the result of (23) into equation (15) for the surface integral over Σ we get

$$\iint_A E \nabla V \cdot d\vec{s} - 4\pi E(P) = 0 \qquad (24)$$

Rearranging terms equation (24) can be written

$$E(P) = \frac{1}{4\pi} \iint_A E \nabla V \cdot d\vec{s} \qquad (25)$$

By vector identity $\nabla V \cdot d\vec{s} = \frac{\partial V}{\partial n}\, ds$ where $\frac{\partial}{\partial n}$ represents a partial derivative with respect to a coordinate perpendicular to the plane boundary surface in a direction out from the enclosed region. Equation (25) can be written without vectors as

$$E(P) = \frac{1}{4\pi} \iint_A E \frac{\partial V}{\partial n}\, ds \qquad (26)$$

Except for the selection of a function V which satisfies all the conditions we have used, equation (26) has the form we require. The left side is just the field at a point P, and since the integral on the right side is on the plane boundary surface A the E in the integrand assumes the given values on the surface. Thus the field at any point P is given in terms of the given values on the surface A by formula (26).

In deriving equation (26) we have imposed restrictions which must be satisfied by the function V. Collecting the requirements to be satisfied by V we have

1. V must have continuous first and second derivations within the region inside the boundaries shown in Figure A-3.
2. $(\nabla^2 + k^2)V = 0$
3. $V = 0$ on boundary surface
4. $\nabla V \neq 0$ on boundary surface
5. $\lim\limits_{\epsilon \to 0} \epsilon^2 V = 0$ at any point P in diffraction region
6. $\lim\limits_{\epsilon \to 0} \epsilon^2 \frac{\partial V}{\partial r} = 1$ at any point P in diffraction region

Fortunately there is a function which meets all these requirements:

$$V = \frac{e^{ikr'}}{r'} - \frac{e^{ikr}}{r} \qquad (27)$$

where r and r' are defined by Figure A-4 as

r = distance from P to any point Q

r' = distance from P' to any point Q

P' = mirror image of P
(i.e., PP' is perpendicular to the boundary A and $d = d'$)

$$k = \frac{\omega}{c} = \frac{2\pi}{\lambda}$$

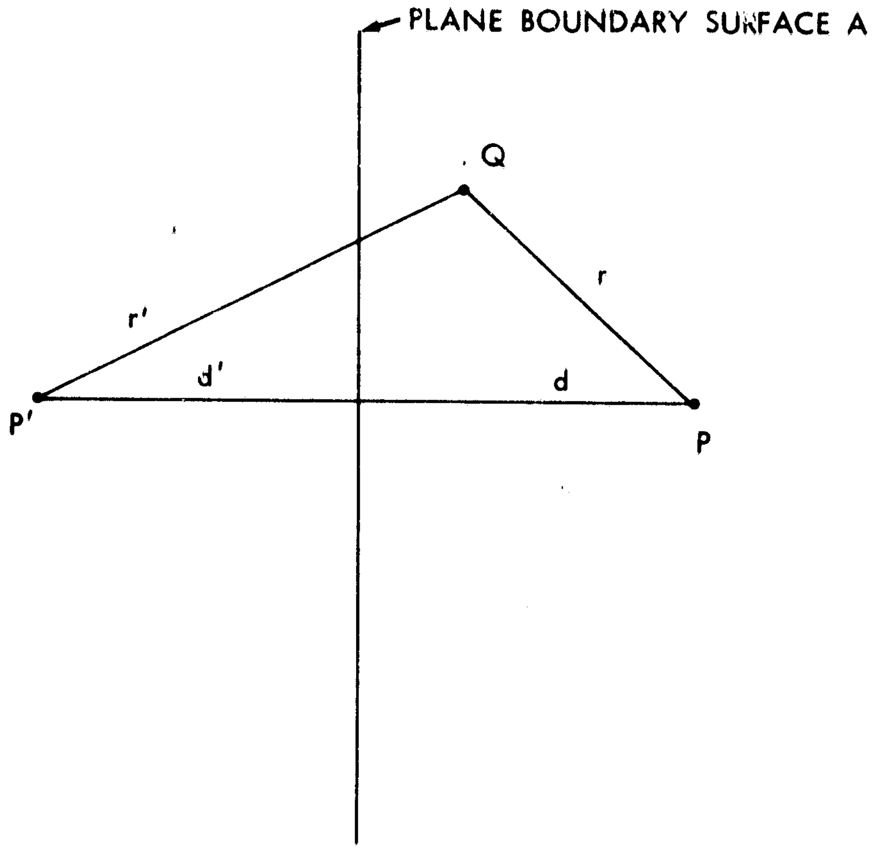

Figure A-4—Geometry for definition of V.

That our function V given by (22) does in fact satisfy all requirements can be proved by direct substitution into the expressions listed above. Here we will only discuss the continuity requirements (item 1 above). As defined by equation (27) V has discontinuities at $r = 0$ and at $r' = 0$. These discontinuities appear at the points P and P' respectively and P' lies outside the diffraction region and P was separated out of the integration region by our sphere Σ. Thus the only points at which discontinuities appear are outside the region specified in the continuity requirement and V given by (27) does satisfy this requirement.

Returning to equation (26) we note that $\frac{\partial V}{\partial n}$ on the surface A is required rather than V itself. We can select coordinates so that the z axis is perpendicular to surface A as shown in Figure A-5 so that $\frac{\partial V}{\partial n}$ becomes $-\frac{\partial V}{\partial z}$ (minus sign appears since the positive Z direction is opposite to the positive $\vec{n}$ direction). From the geometry of Figure A-5, r and r' are given as

$$r = \left[x^2 + y^2 + (d - z)^2\right]^{1/2} \qquad (28)$$

$$r' = \left[x^2 + y^2 + (d + z)^2\right]^{1/2} \qquad (29)$$

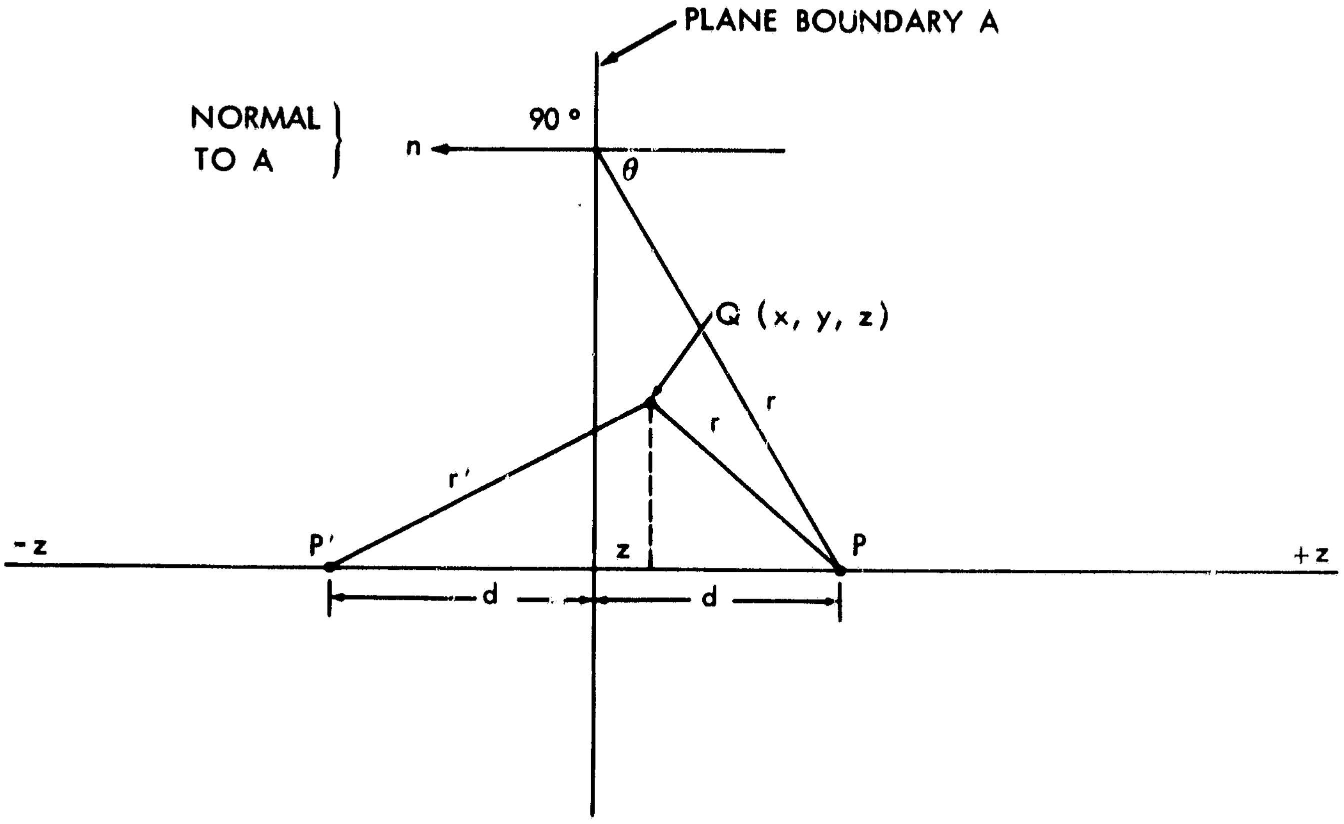

Figure A-5—Geometry for definition of terms in V.

Substituting (28) and (29) into (27) we can find

$$\frac{\partial V}{\partial n} = -\frac{\partial V}{\partial z} = -\frac{\partial}{\partial z}\left\{\frac{e^{ik\left[x^2+y^2+(d+z)^2\right]^{1/2}}}{\left[x^2+y^2+(d+z)^2\right]^{1/2}} - \frac{e^{ik\left[x^2+y^2+(d-z)^2\right]^{1/2}}}{\left[x^2+y^2+(d-z)^2\right]^{1/2}}\right\}$$

$$= \left\{\frac{-ik(d+z)}{\left[x^2+y^2+(d+z)^2\right]} + \frac{d+z}{\left[x^2+y^2+(d+z)^2\right]^{3/2}}\right\} e^{ik\left[x^2+y^2+(d+z)^2\right]^{1/2}}$$

$$+\left\{\frac{-ik(d-z)}{\left[x^2+y^2+(d-z)^2\right]} + \frac{d-z}{\left[x^2+y^2+(d-z)^2\right]^{3/2}}\right\} e^{ik\left[x^2+y^2+(d-z)^2\right]^{1/2}} \qquad (30)$$

Now in equation (26) we are integrating over the surface A so that we want the value of $-\frac{\partial V}{\partial z}$ on A which is obtained by setting $z = 0$ in equation (30)

$$\left.\frac{\partial V}{\partial n}\right|_A = -\left.\frac{\partial V}{\partial z}\right|_{z=0} = \frac{-2e^{ik\left[x^2+y^2+d^2\right]^{1/2}}}{\left[x^2+y^2+d^2\right]^{1/2}} \frac{d}{\left[x^2+y^2+d^2\right]^{1/2}} \left[ik - \frac{1}{\left[x^2+y^2+d^2\right]^{1/2}}\right] \qquad (31)$$

Equation (31) can be simplified by noting from equation (28) and the geometry of Figure A-5*

$$r = \left[x^2+y^2+d^2\right]^{1/2} \quad \text{when} \quad z = 0 \qquad (32)$$

$$\cos\theta = \frac{d}{\left[x^2+y^2+d^2\right]^{1/2}} \qquad (33)$$

Substituting (32) and (33) into the equation (31) we obtain

$$\left.\frac{\partial V}{\partial n}\right|_A = -\left.\frac{\partial V}{\partial z}\right|_{z=0} = \frac{-2e^{ikr}}{r} \cos\theta\left[ik - \frac{1}{r}\right] \qquad (34)$$

Now we can substitute (34) into (26) and complete our diffraction formula

$$E(P) = \frac{-1}{2\pi}\iint_A E_A \frac{e^{ikr}}{r} \cos\theta\left[ik - \frac{1}{r}\right] ds \qquad (35)$$

where (refer to Figure A-5)

E(P) = Electric field at a point P in the diffraction region.

E_A = electric field on the plane boundary A.

r = distance from P to a point on A.

θ = angle between r and normal to plane A.

$k = \frac{\omega}{c} = \frac{2\pi}{\lambda}$

*Note that in Figure A-5, P and P′ were chosen as point on the z axis to obtain equation (32) and (33). In general x would be replaced by $(x - x_0)$ and y would be replaced by $(y - y_0)$ where x_0, y_0 define the x, y coordinates of the points P and P′. In the derivation the z coordinate of the point P was represented by d to avoid confusion with the coordinates of the point Q. In general the d in equations (32) and (33) is replaced by z. Otherwise the general results have the same form as found above.

Given the electric field E at every point on the surface A, equation (35) can be used to determine the electric field E(P) at any point P in the diffraction region. This statement depends, of course, on whether the integration indicated in equation (35) can be performed. If the integration cannot be performed analytically, it can be assumed that a numerical solution to any desired accuracy can be obtained using a computer. In many problems of interest, satisfactory results can be obtained by approximating equation (35) using the geometry of the specific problem. For example, for small angles θ such that $\cos\theta \simeq 1$ and at great distances r such that $1/r \ll k$, we can approximate equation (35) as

$$E(P) = \frac{1}{i\lambda} \iint_A E_A \frac{e^{ikr}}{r}\, ds \tag{36}$$

Equation (36) represents Huygens' principle since the contribution from each point on the boundary surface is given by $E_A\, e^{ikr}/r$ which describes the spatial variation of a spherical wave. Thus the diffracted field as given by equation (36) can be interpreted as the summation (integral) of spherical waves radiating from each point on the diffraction boundary.